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PREFACE

I must point out that this is not an essay on science philosophy but a personal reflection on some fundamental notions and concepts underlying the theory of phase transitions, such as limiting processes, all parts of our classical mathematical heritage. Specifically, this essay is devoted to commenting and analyzing the study of many particle Bose systems in the framework of the phase transitions theory of infinitely extended systems.

Currently, experimental progress in the study of mesoscopic particle systems has led to a better understanding of phenomena associated to finite particle systems, opening conceptions such as; thermodynamic limit, breaking of symmetries, phase transitions, to criticism and permanent revision.

Often we confuse the phenomena with our attempts for representing them through mathematical or conceptual models. Indeed, notions such as limit and continuity like to be effective ways to approximate the discrete nature of many physical phenomena. The transitions between different thermodynamic states of a many particle physical system exist, as well as the ruptures of symmetries. However the identification of the singularities of thermodynamic functions, which arise in the thermodynamic limit-never reached in practice-, with the cause of the phenomenon, seems to be a debatable matter, although in their vicinities the probability of the transition occurring is very high and in fact it should take place. In other words, the probability of the system accessing different states increases significantly when the values of certain parameters are in the vicinity of a singularity of the limit functions. Moreover, surely we will never mathematically describe such kind of phenomena in such a way that the underlying models can give us absolute and exhaustive physical description about them, but we can always predict its occurrence with a degree of certainty appropriate to our purposes.

The first five chapters concern with topics such as, realism, relativism, the role played by mathematics in the development of physical theories, the problems of underdetermination of scientific theories and the demarcation between science and non-science, among others.

The remaining chapters deal with the main mathematical strategies used for studying bosonic phase transitions. A very important place in the framework of the spontaneous rupture of continuous symmetries is occupied by the notion of quasimeans developed by N. N. Bogoliubov. We will also briefly analyze the open questions left by the results obtained from experiments with trapped gases (mesoscopic systems).

The validity of our physical models depends on their correspondence with reality, even though this may not be immediate. This is the case of the so-called non conventional Bose-Einstein condensation (independent on temperature), discussed in chapter 20. Thus, for example, the ideal Bose gas disturbed by a suitable external source, breaking the symmetry associated with the conservation of the number of particles, undergoes, in the thermodynamic limit, a macroscopic occupation of the ground state not related to any singularity. Moreover, this is an independent on the temperature occupation. In this context, I included in chapter 21 a recent work with Rosanna Tabilo, dealing with this interesting subject.

Contents

1. Introduction

There is no dissimilarity between the human act of observing an object or experimenting on it and a blind action exerted by one arbitrary physical system on another. The only difference, is that, in the first case, it seems to us that the reasoning, free will, and what we call perception mediate such action. That is, it materializes due to a cause or reason that we consider "not blind." However, for all practical purposes, the only thing that underlies this phenomenon, in both situations, is the "interaction" between two objects of the world, one of which we arbitrarily call "subject" because we recognize ourselves in it. There is no any other way in which the materiality of things manifests its existence to the human species.

An experiment is a physical process like any other. "Tunnel effect", "superconductivity","superfluidity" or "bosonic condensation", to mention some phenomena, are not a product of human "subjectivity." Indeed, they do not require the presence of any observer to occur.

If we stop looking at the Moon, all the effects that its objective presence produces on us, directly or indirectly, will not be eliminated. Thus, the world is manifested and shown through interaction.

In this physical encounter, which in many ways is a collision, we recognize the objects of reality, beyond any name or denomination with which we identify them. Their existences become evident to us when we transform them into tools, extensions of our our sensory organs devoted to intervening and changing the world.

Here, there is nothing to prove. The recognition of this basic facts is, finally, foundation of the materialistic realism that this essay deals with. The same realism that makes the chinese philosopher Mo-Tzu, in the fifth century BC, to affirm the existential primacy of the things of the world above the names we assign them.

On the other hand, the systematization, application and extension of this knowledge of the reality, product of this fundamental and material interaction, are possible in the fertile ground of culture, built on complex networks and links. If our species wishes to “know” the world, it must go to meet it.

Thus, what is impossible for a man is feasible for the species. And this common enterprise is achievable because we do not differentiate one from the other so significantly as to make the phenomenon of communication unviable, even though we belong to different contexts. Our evolution has been structured on this fact.

The paths that led humanity to what we currently know as science include the establishment of criteria of truth intended to demonstrate conjectures or propositions, in a scenario of experimental verification. Thus, slowly, a set of novel practices and strategies were set up, freeing the emerging scientific knowledge from the predominant scholasticism in the later Middle Ages.

Since the Renaissance and the advent of rational empirical science, man and nature are in an unprecedented symbiosis in human history. From this moment, observation, experimentation and reason are at the service of each other in a constant flow.

The results of scientific research will depend not on the desires or simple intuitions of homo sapiens, but on experimentation and on the very nature of the phenomena, that is on factors associated with their intrinsic characteristics.

In this sense, Galileo, when analyzing the speed of fall of the bodies, leaves the Aristotelian teleology. He stops worrying about the final causes, looking for the regularities of phenomena - "accidents" - and introduces mathematics as an indispensable instrument for natural philosophy. Thus, the long path that culminates with the advent of rational empirical science is a by-product of human evolution as a whole. This is the recent history of a liberation, initiated from the very moment when Galileo dismisses some of the thesis of the aristotelian philosophy.

Great conflicts, changes or revolutions are not the product of immediacy, but the result of long historical processes. Despite differences, emphases and positions, the evolution of what concludes today with what we call science is the history of a gradual but persistent process of decanting towards more independent styles of thought. The advance to the knowledge of the nature supposes to the human being to resign an important part of its magical vision of the world, establishing a more distant connection with it.

The spectacular development of group theory and geometry from Gauss, Galois, Klein, Cauchy, Lobachevski, Riemann and other mathematicians during the nineteenth century, along with notable discoveries in physics, led, during the first quarter of the 20th century, to the the emergence of the two most important theories of the last hundred years: the theory of relativity (special and general) and quantum mechanics. With them, in addition, there is an explicit rupture with previous conceptions of reality.

Just as contingency defined the position of Fourier, Laplace, and Galois during or immediately after the French Revolution, or that of the natural philosopher Isaac Newton (1643-1727), his disciple Colin Maclaurin and his contemporary Robert Boyle - impregnated with the protestant faith-the philosophical and/or ideological positions of those who were actors in the epoch that preceded and followed the aforementioned scientific revolutions (quantum mechanics and relativity) -Hertz, Mach, Bohr, Pauli, Heisenberg, Bohm, Einstein, Schrödinger-are part of historical circumstances, which is that of two world conflagrations, a product of, among other things, the expansionist policies of European economic empires, the impoverishment of large sectors of society and the maintenance of feudal structures (the old rural structure persisted in France until 1914), etc.

What defines us as human beings is the full assumption of our freedom, not of our circumstance, but of the position we assume in front of it. In deciding we not only decide for ourselves but for all human beings. Such positioning is always an act even if it acquires the form of disaffection or lack of interest. Thus, our existence will ultimately be nothing more than the sum of those moments. In the same way, all transcendence is only memory collected by others.

Thus, the Copenhagen school, part of the group of scientists who generated the quantum theory, has an ideological debt with the classical German philosophy, especially with Kant, with the positivism, and extreme empiricism.

> But why then had Born not told me of this "pilot wave". If only to point out what was wrong with it? Why did Von Neumann not consider it? More extraordinarily, why did people go on producing impossibility" proofs, after 1952, and as recently as 1978? When even Pauli, Rosenfeld, and Heisenberg, could produce no more devastating criticism of Bohm's version than to brand it as "metaphysical" and "ideological"? Why is the pilot wave picture ignored in text books? Should it not be taught, not as only way, but as an antidote against the prevailing complacency? To show that vagueness, subjectivity, and indeterminism, are not forced on us by experimental facts, but by deliberate theoretical choice?. (J. Bell [149])

Although I have pointed out names to indicate important milestones, it seems necessary to emphasize that science is not only the final result of contributions from individuals. Science and scientists are part of the human history development.

This activity is a by-product of that interminable movement, in which the whole seems to prevail over the individual, and the new knowledge necessarily emerges from the previous.

Many of the issues that will be open to discussion in the light of the apparent or real conflict between quantum mechanics and relativity (at least in the so-called Copenhagen interpretation) are nothing more than a resurgence of very old disputes), present even today, for example, regarding determinism, indeterminism, causality, chance, etc.

Independently of the legitimate position that each one can take with respect to the future of each one of those theories, the clarification of supposed incompatibilities, if any, will, surely, be the responsibility of the next generations.

The conflict Kuhn-School of Vienna or Kuhn-Popper, previous to the Congress of Philosophy of Science (1965-London) where both characters faced, is not the one of the realism versus the idealism, but the manifestation of a conflict generated in the prevailing functionalism, where logicism, cryptic and abstruse for the general public and even for the academic world, Popperian falsifiability, the concept of "scientific programs", due to Imre Lakatos, and Feyerabend's anarchism, will confront each other and to Kuhn's psychologism with his "History of Scientific Revolutions," a quick-read, easily digestible text.

Whether or not Kuhn intended his work to end up as a useful instrument of relativism and anti-realism, it ended up being its destiny.

Henceforth, with the abandonment of any possibility of determining, epistemically speaking, the validity or superiority of any of the so called "scientific paradigms," concepts such as "conceptual framework", "evaluation standards" and "consensual truth" were made part of the lexicon and practice of many scientists and philosophers. Thus, necessarily, it is placed in the best tradition of "pluralist" or "universalist" constructivism and, as a corollary, its proposal is established as the preferred weapon of the relativism defended by postmodernism.

A set of *a priori* assumptions, however plausible they may seem, and *a posteriori* conclusions, rooted in a logical and/or mathematical language do not constitute necessarily a science. Mathematical models are instruments, tools, intended primarily to describe and allow the predictability of natural phenomena and not to transform themselves into realities in a pythagorean sense. A direct reference to reality either through conducted experiments, or verifiable facts will be always needed.

These assumptions-which one might call metaphysical ones-are otherwise necessary. Without them, the advent of science would have been impossible. This adventure, from Galileo but mainly from Newton to our days, is that of scientific realism, which implies, in addition to the acceptance of the existence of a knowable external world, the gradual and consistent abandonment of the atavistic practices and beliefs who ruled the Western world for millennia. I say "Western" because, since the ancient Greece, our science is, indeed, the product of the historical development of the West.

Today, the exacerbated "subjectivisms", "relativisms" and "functionalisms" have deployed in their fullness their non aseptic essences, by contributing, directly or indirectly, to the replacement, in general, of the real for a consciousness, if not false, at least distorted of our world. In that sense, if we only aspire to a relational science (like that of structural realism) or a phenomenal science, we will be content to ask ourselves "how?"

However, that suspension of other fundamental questions that in sciences such as physics may be temporary legitimate, becomes unacceptable in social sciences, because these are precisely the answers to "what for?" and "why?" those that allow us to intervene the social reality of man, his own creation.

Man has deciphered his own genetic code, intervened and modified genetics of bacteria by creating new species, designed particle accelerators, powerful enough to scrutinize the microworld of elemental particles emerging from quantum vacuum (Higgs bosons, neutrinos, etc.), has generated important mathematical and physical theories (string theory, standard model, among others) to account for all physical phenomena (gravitation, quantum electrodynamics, etc.), always to the saga of a hitherto elusive unified theory. He has taken important steps to unravel the mysteries of the universe. Human society has more than enough technical means to handle huge amounts of information. Technology allows us to observe the cosmos to previously unsuspected limits (observatories or radio telescopes), and to handle information, extending our senses beyond the imaginable.

On the other hand, despite the current technological development, the profound transformations in terms of our world view and social implications that should have been produced by the effect of scientific activity are far from being materialized in a global and definitive sense. Quite the contrary, what dominates in our time is the discourse of a legion of necromancers, alchemists, saints, healers and prophets of the apocalypse. They no longer travel through narrow alleyways of small towns or isolated villages where the sounds are only echoes of the footsteps of their own few inhabitants. Nowadays they wander through the open spaces and infinite avenues of the great cities, vociferating their omens, crying out to ghosts, using the media, carrying their loudspeakers, appealing to those irrational fears coined in our customs and brains in past times. Thus, what prevails today is pseudoscience.

Scientific knowledge has become cryptic, disintegrating and impoverishing itself immediately to the contact with the asphalt of cities, like a distant rumor that decays to be only a set of disconnected, nonsense and distorted sentences.The promised knowledge society has not come true yet. The epicurean plan to free human beings from their fears is truncated.

2. Some preliminary reflections

2.1. Science and underdetermination. American theoretical physicist Kip Thorne in his book "Black Holes and Time Warps" wrote,

> In the curved spacetime paradigm, the verbal picture of Einstein's field equation is the statement that "mass generates curvature of the spacetime." When translated into the language of the flat space paradigm, this field equation is described by the verbal picture "mass generates the gravitational field that governs the shrinkage of rulers and the ticking of clocks." Although the two versions of the Einstein's field equation are mathematically equivalent, they verbal pictures differ profoundly.
>
> It is extremely useful in relativity research, to have both paradigms at one's fingertips. Some problems are solved most easily and quickly using the curved spacetime paradigm; others using flat spacetime. [129].

It is not surprising that two different perspectives, such as those described by Thorne, result to be mathematically equivalent, considering that they attempt to describe a same phenomenon. However, this does not imply that the purpose of elucidating which of them gives a better representation of reality should be abandoned by science. Thus, for example, classical field theory (CFT) allowed to overcome the non-locality of Newton's theory, a problem recognized by him, regardless of the new questions arising from the new theory.

That the scientific proposals elaborated to make the world comprehensible to us are not complete and need the analysis and development of new conjectures and hypotheses is not new.

Scientific activity is constructed in that way. Surrender to fatigue or relativisms, including pragmatism, would be a fatal mistake.

The question pointed out by Thorne is what has been denominated "underdetermination of scientific theories," that is, phenomena are explicable, empirically, by more than one theory.

The so-called "conventionalist" current, in general, considers that theories finally depend on customs and conventions, which determine only "provisional trues." This fact inscribes this position in the class of relativisms.

Henry Poincaré in his article entitled "The Space" (1902), published in Science and Hypothesis, refers to beings living on a sphere and given themselves to the task of determining the geometry of their world, without noticing that the temperature in this solid body is variable and that it descends uniformly from its center to the surface. The final effect would be the dilation of the objects when approaching the center and its reduction in size, including the measuring instruments, when approaching the surface. His conclusion was the following one:

> If to us geometry is only the study of the laws according to which invariable solids move, to these imaginary beings it will be the study of the laws of motion of solids deformed by the differences of temperature alluded to [...] If they construct a geometry, it will not be like ours, which is the study of the movements of our invariable solids; it will be the study of the changes of position which they will have thus distinguished, and will be "non-Euclidean displacements," and this will be non-Euclidean geometry. So that beings like ourselves, educated in such a world, will not have the same geometry as ours.[130]

If the question is that the geometry we determine will depend on our special way of interacting physically with our environment, which would make it impossible for us to establish whether it is intrinsic and structural to the universe - if that can make sense -, I must agree on this point with Poincaré.

He indirectly assumes, when posing the problem, a essential question: the possibility of mathematizing reality. This is a fundamental issue to take into consideration. On the other hand, although we will never have a view of our physical world independent of our models, as Stephen Hawking proposes, they do not have to be a pale image of physical realm, in other case we would be condemned to inhabit the cave of Plato, from which we will never leave.

It seems necessary to highlight some aspects of the problem that I consider important. First, science, in this case physics, can not determine definitive limitations to its own knowledge of world, even if they appear as inevitable, since under these conditions, progress becomes a difficult task. Otherwise, we had to settle for ptolemaic cosmology. Indeed, nowadays, it should be premature to think of the existence of unbridgeable limits of physical knowledge and unaffordable scientific questions, unless we declare ourselves hopelessly skeptical.

It would be arrogant to think that science must abandon purposes that it surely shared, in its beginnings, with philosophy. The first objective is to assume the possibility of accessing to knowledge and constituting itself as a transforming instrument of the world.

The second purpose is the determination of those elements that constitute reality, which inevitably refers us to the ontological question.We are beings for action, for praxis, but this does not prevent us to assume the before mentioned task.

This purpose, apparently resigned by some groups of the scientific community, has been assumed by philosophy. In the paper "On the epistemological implications of geometric convention", published in Ontology Studies (2012), regarding the geometric questions mentioned by Poincaré and outlined very briefly in this essay, the author writes:

> If the geometry of the world is the result of convention, agreement and comfort (whether conceptual or heuristic), it seems almost impossible to access the true nature of the cosmos. However, this finding does not seem to imply a renunciation of their understanding. Perhaps a revision of the concept of intelligibility, insofar as one accedes to the world itself, loses its intuitive sense and retreats to the pretension of regularity and predictability. [131]

2.2. **Ceteris paribus.** It is clear that a set of axioms, finite in addition, is a starting point to construct a narrative that can give not only logical but also mathematical explanatory coherence to the regularities that we observe. That fact makes impossible for us to identify in an absolute way these postulates with the detected or conjectured uniformities since they possess only a conditional and therefore provisional character.

For example, the law of Newtonian inertia turns out to be an idealization. In practice all objects of world interact with gravitational fields, even weakly. From this it can be concluded that there are no bodies free from external forces. In this sense, identifying our theories with the reality of the world, in a sense of one-to-one correspondence, is wrong. Thus, axioms, principles or postulates, especially when set in a context of universality, have to be always under scrutiny. In this sense, it is likely that from our most successful physical theories, relativity and quantum mechanics, little will remain in the future. Possibly their fate will be to be replaced by broader theories that somehow include them.

On the other hand, it should be arbitrary to infer from the absence of absolute correspondence that scientific theories do not reflect more than our personal perceptions of the world and say nothing of what it is.

Rational empirical science is an activity that has become highly complex, both in its theoretical and practical development.

The permanent discussion of matters such as “demarcation”- or how to distinguish science from non science- and “underdetermination” account for this fact, even though these problems seem to be of greater interest for philosophy than for science itself.

On the other hand, that certain variables, for example in a mathematical model, can be considered to be non-influential, constants or dispensable, is a natural and necessary practice in science. This is the so-called “Ceteris Paribus” clause - all other things being equal - naturally incorporated into all science.

Physicist Lisa Randall mentions in his work “Warped Passages. Unraveling the Mysteries of the Universe’s Hidden Dimensions” [34] an essential aspect of all science: the necessary separation between the meaningful and that accessory for the understanding of a phenomenon.

> Scientists often average over or ignore (often unwittingly) physical processes that occur on immeasurably small scales when formulating their theories or setting up their calculations. Newton’s laws of motion work at the distances and speed she could observe. He didn’t need the details of general relativity to make successful predictions. When biologists study a cell, they don’t need to know about quarks inside the proton.[132]

Thus, although contingency is one, indivisible, our brains discriminate those aspects of reality that are indispensable for us to act, fragmenting and making it apprehensible, reducible and finally manageable. However complex or strange the world may seem, its intelligibility lies precisely in the possibility of performing this operation. The inapprehensible and chaotic world exists only in the minds of those who wish to see that.

3. Mathematics

Nowadays, almost every scientific and technological discipline makes use of mathematical advances, however, mathematics itself has become independent of its initial roots. The degrees of generality and abstraction of mathematical theories have surpassed all expectation and the underlying creative process resembles us that of the artistic creativity.

Rodolfo Llinás pointed out that [157],

> For an organism to interact successfully with its external world, its nervous system must be able to handle (processing and understanding) easily and quickly the signals that it receives through its senses. Once this information is transformed into a suitable executed motor signal, it is returned to the external world. It is obvious that the properties of the internal functional space and those of the external world are different, however for the motor response to have some useful meaning, they must have some similarity between them. This internal functional space constituted by neurons, must represent the external world properties and, in some sense, they must have equivalent attributes. Such as a translator must operate with conceptual continuity between the languages he is translating, this internal functional space must preserve the conceptual continuity.

In other words, it would be an absurd to think that our own biology, in permanent interaction with the environment, does not give us adequate information about it. In this way, human beings possess innate or, if you will, biological abilities to identify patterns, quantities (such as the so-called "mumerosity", determined by topographic maps located in the parietal lobes of the brain - giving the neuronal basis to our arithmetic capabilities-), recognition of space, abstraction capacity, etc., that is, capacities structured on our own material nature, without whose contribution would have resulted impossible what we call mathematical reasoning, based on the generation of symbolic representation and fundamentally on the presence of language, a phenomenon that would have emerged prior to the development of numerical systems, based on social interaction, that is, as part of the culture [158-159]. In this same direction, currently, active research areas are the study of the neural activity associated with the generation of conclusions in deductive problems of an elementary nature.

In any case, although the human capacity to develop mathematics depends on the brain, it emerges from the interaction of man with nature and his social environment as a historical product. Thus, although mathematics satisfies all the requirements of a constructive social process, the biological imprint that underlies its emergence is undeniable, being both complementary aspects of its development. Thus, the intuitionist-logicist dispute seems to have always been futile.

Finally, the multiple attempts to categorize and elaborate a kind of taxonomy of the processes implicit in the development of scientific research, whatever the discipline, seem to be diluted because of the complexity and diversity of ways in which this activity manifests itself. This should not be considered as a disadvantage but as a circumstance that favors its development and advance, considering that diversity is, precisely, one of the pillars on which our own evolution is based.

4. Neither substantialism nor relationalism

There are not, in nature, such things as circles, triangles, straight lines, dots, etc, however these basic abstractions allow us to handle reasonably well our every day world. What we hope is that the qualitative and quantitative differences between our idealizations and approximations (these notions are different but not mutually exclusive) to the physical realm turn out to be negligible compared with it, in any proper sense. This fact should not be understood as the resignation to the legitimate aspiration of elucidating the significant factors that account for the materialization of any phenomenom, restricting science to a merely descriptive or phenomenological role. Thus, the substitution of the old metaphysical assumptions by the logical coherence of mathematical constructs, does not relieve us of our obligation of giving a satisfactory physical explanation (despite the controversial nature of this notion from a philosophical point of view) about the natural phenomena, that is to say, in a substantialist or realist way. In other words we should

> [...] not ignore or leave out of account altogether the details of the mechanisms, whatever it is, that is in operation in the phenomena under discussion [151]

Therefore, the success of relational theories does not justify giving up the search for the mechanisms underlying natural phenomena.

In this scenario, the role of mathematical physics has been, historically, to give disciplinary support to many of the achievements of physicists. However, the relationship between both sciences could be described as *bittersweet* in many cases. In this sense, Barry Simon declares:

> The analysis of mathematical models for physical phenomena is part of the subject matter of mathematical physics. By analysis is meant both the rigorous derivation of explicit formulas and the investigation of the internal mathematical structure of the models. In both cases, the mathematical problems which arise lead to more general mathematical questions not associated with any particular model. Although these general questions are sometimes problems in pure mathematics, they are usually classified as mathematical physics since they arise from problems in physics [...] The techniques used and the general approach to the subject have become more abstract. Although in some areas the physics is so well understood that the problems are exercises in pure mathematics, there are other areas where neither the physics nor the mathematical models are well understood. These developments have had various serious effects not the least of which is the difficulty of communication between mathematicians and physicists. Physicists are often dismayed at the breadth of background and increasing mathematical sophistication which are required to understand the models. Mathematicians are often frustrated by their own inability to understand the physics and the inability of the physicists to formulate the problems in a way that mathematicians can understand [152].

For Erwin Schrödinger

> The demand for continuous description was encouraged by the fact that the mathematician claims to be able to indicate simple continuous descriptions of some of his simple mental constructions [...] Physical dependences can always be approximated by this simple kind of functions (the mathematician calls them 'analytical', which means something like 'they can be analysed'). But to assume that physical dependence is of this simple type, is a bold epistemological step, and probably an inadmissible step [141].

It would be a mistake to think that a physical conjecture can end up settling as a valid or a plausible one only because it has be founded on rigorous mathematical notions. In this sense, it is expected that the consistency and explanatory power of the physical meaning behind all mathematical discourse prevail over any other considerations, including those of an aesthetic nature (beauty or simplicity of mathematical constructs, for example). Moreover, it seems an unreasonable idea to discard a theory before evaluating its theoretical soundness against the experiments or the appearance of new phenomena.

The relational character of mathematics implies that two different physical conjectures (even contradictory between them) about reality may lead to the use of similar analytical tools, especially in the sense that they must account for a unique experience, backed by data and measures also unique.

Logic and therefore mathematics do not care about the verbal sense of propositions. Indeed, it is a well known fact that a hypothesis will be proved false when it leads to a false conclusion (in other words if conclusion is false, then the hypothesis must be false).

> Science uses tested methods to filter the facts. This normative element is of course most visible in the psychology of the mathematician, for even though his thoughts are always in some sense "correct," there is a psychological distinction to be made between what he knows to be true and what he simply conjectures or intuits. But a normative component can also be sensed in the essentially organic conception of things that encrusts logical thought in the world. In any case, experimental testing always begins what one believes to be logical. Hence the failure of an experiment sooner or later entails a change of logic, a deep change in our thinking. (G. Bachelard, [120])

In this sense, until its experimental or observational refutation, a false physical conjecture, accompanied by a mathematically coherent theory, might seem to us true. Thus, for example, violations of the conjectured parity symmetry (P) as well as charge-parity symmetry (CP) introduced by E. Weyl in quantum mechanics in 1927 and 1932 were experimentally detected in the cases of the weak interactions in 1957 [116] and 1964 [117], respectively. Before the first experiment, P and CP symmetries were assumed true for all interactions in nature.

> I felt sure that parity would not be violated, there was a possibility it would be, and it was important to find out.(Richard Feynman [118])
>
> A rather complete theoretical structure has been shattered at the base and we are not sure how the pieces will be put together. (I. Rabi [119])

This suggests us that it is risky to assume the universality of intuitive physical notions when considered in the context of counterintuitive theories, such as quantum mechanics.

The mathematical tools used almost one thousand five hundred years ago to geometrically determine the trajectories of the celestial bodies under the geocentric assumption *the earth is flat and fixed and all the objects in the known universe rotate around it*, surely produced analogous results, restricting ourselves to the ancient scenario, to those obtained nowadays by using the notion of relative motion. Moreover, two plausible physical conjectures could lead to similar mathematical models. That means that for their construction, such assumptions are sufficient but not necessary conditions. On the other hand, the same phenomenon can manifest different traits, depending on the context. This makes it necessary to apply different mathematical strategies in each case. The above mentioned facts suggest us that there is a clear risk of overestimating the role of our mathematical idealizations as support of our physical theory, that is to say, conjectures.

Note that many mathematical models or complex mathematical constructions serve to relate data and experiments without necessarily being consubstantial to a particular hypothesis or that is to say, without having been derived as sine qua non conditions from it or from basic principles or even without considering the mechanisms underlying a physical phenomena. In this sense, these models are not explicative but heuristic tools which could or not be useful for elucidating mechanisms. Thus, for example, although the renormalization group theory was built on the basis of mathematical considerations it gives valuable information about phenomena exhibiting universality in the sense that they can be grouped, with their help, in classes associated to global features.

Finally, it has to be pointed out that the use of heuristic mathematical tools as, for example, Dirac functions, Feynman path integrals or integrals over spaces of paths (quantum field theory), symmetry breaking terms in classical and quantum many particle systems (statistical mechanics), etc., has represented usually a necessary strategy to the study of nature. In spite of sometimes that tricks lead, finally, to well defined mathematical notions, which may give some theoretical support to physical conjectures, this fact does not constitutes by itself a sufficient condition to validate them. Thus, the so-called string theory, theoretical construction which includes the notion of curved hidden dimensions introduced by the Kaluza-Klein theory (Kaluza-Klein compactification of extra dimensions) [121], far away of being physically confirmed yet, has led to arduous disputes between defenders [122] and detractors [123]. In this case, the lack of empirical evidence (after forty years of research), one of the main criticisms to this theory, seems to be a little unfair in the sense that many physical conjectures have been experimentally tested only after many decades (for example Bose Einstein Condensation). Since this proposal is still far from being validated, perhaps the most important objection that can be made is related, not to the theory itself, but mainly to an overly triumphalist media treatment to which some of its own creators have possibly cooperated because of an excess of enthusiasm.

5. Phase transitions

Allow me a small digression on infinite limits. In spite of the ancient greek fear of infinity, many of the current physical theories are founded upon mathematical concepts such as infinite processes. As well as the method of exhaustion based on sucessive aproximations, developed by Antiphon and Bryso and whose correct mathematical proof is due to Eudoxus, was used in ancient times to calculate areas of different geometric figures [42], notions as *thermodynamic limit* and *infinitely extended systems* have been commonly applied in the framework of equilibrium statistical mechanics of classical and quantum systems to study real physical systems. Unlike Eudoxus, who consciously avoided to prove his hypothesis by successive approximations which involves considering the underlying limits, today, infinite processes are part of the usual language of those scientific branches.

In classical and quantum equilibrium statistical mechanics, phase transitions are associated to the non analyticity, of well known thermodynamic functions. H. Kramer in 1934 [1] was the first in suggesting that the above mentioned thermodynamic limit, consisting in that the density of particles remains constant while both the number of particles and the volume of the region enclosing them tend to infinity in a suitable way, should be used to explain singularities of the mentioned functions. This limit was introduced, in practice, by the first time by Lars Onsager in 1944 [2], for exactly solving the two dimensional Ising model, proving by this approach the existence of different phases. In this sense, phases correspond to zones of analyticity of the limit free energy, while phase boundaries, are identified with its regions of non-analyticity.

Mathematical aspects of the theory of phase transitions based on this strategy were initially developed by van Hove in 1949 [3] and N.N. Bogoliubov in 1946 [4] and received further impulse from works of many mathematicians and physicists during sixties and seventies of the past century (see for example [8,20,27,39], [59-62], [69-72], [77,79,80], [86-91]).

Since, all macroscopic quantities of physical interest can be obtained from different order derivatives of basic thermodynamic functions as the free energy, Paul Ehrenfest in 1933 [6], classified phase transitions by using the lowest order of some derivative of the free energy presenting discontinuities. Thus, first order phase (or sudden transitions) and second order transitions were associated to discontinuities in the first and second derivatives of this function, respectively. However this categorization admits to be extended to high order phase transitions.

Therefore, they have been interpreted, mathematically speaking, as singularities of thermodynamic functions in the thermodynamic limit. On this conception lies not only Landau theory of continuous phase transitions, but also Lee Yang and renormalization theories, including all mean field strategies developed the last century.

In spite of the fact that far away from a small vicinity of a critical point, where fluctuations can be ignored, the so-called infinite equilibrium states (or KMS states in the quantum case) approximate fairly well both qualitatively and quantitatively the thermodynamic behavior of many real finite systems, the finite thermodynamic functions do not exhibit any discontinuity or singularity and in this sense phase changes cannot be derived from them. Moreover, this approach is unsatisfactory for the study of critical points, where subtle microscopic phenomena like fluctuations appear to contribute, facilitate or even produce phase transitions. On the other hand, the renormalization group theory (RGP) has given fruitful phenomenological insight in some cases and it has failed in other situations. In this context it has long been known that the standard definition of phase transitions does not apply to all cases and it should be generalized or reformulated independently of notions such as thermodynamic limit. Boltzmann directly poses the problem [58]:

> Which represents the observed properties of the matter more accurately, the properties on the assumption of an extremely large finite number of particles, or the limit of the properties if the number grows infinitely large?

Oliver Penrose in 1970 [97] pointed out that the real problem of statistical mechanics can be reduced to find and use the relationship between the macroscopic description (coarse-grain scheme) of physical objects and the microscopic description of them (fine-grain scheme) as dynamical systems of molecules.

In the last ten years these issues have been subject to a great deal of critical examination by science philosophers [48-52].

The question arises whether this is the best that can be done.

Interesting attempts to avoid the use of thermodynamic limit, in a general framework, have been done the last decades. Some of them are related to the definition of a new microcanonical equilibrium state for quantum systems with finite dimensional state spaces whose density of states is a continuous function of the energy, introducing non analyticities at this level [43,44]. Since certain singularities in the entropy of a kind of short range systems have their origin at the microscopic level, it has being suggested that some vestiges of this connection should be found there. On the other hand, the study of the integer quantum Hall effect, discovered in 1980 [63], led to the knowledge of a new type of phase transitions independent on spontaneous symmetry breaking, the so-called topological phase transitions, phenomena quite different from those previously analized. This fact gave further impulse to the QPTT (see for example [46,64] and [65, 66] for criticisms and responses).

However, the best strategy seems to be the use of the so-called canonical ensemble [93] as it shall be shown in other section, at least in the case of real Bose finite particle systems. Moreover, nowadays, the great problem is to characterize phase transitions at a mesoscopic level, i.e., for small particle systems far away from thermodynamic limit.

Let us remember that in 1902, W. Gibbs introduced, simultaneously, the conceptions of microscopic ensemble, canonical ensemble, and grand canonical ensemble.

> The problem of why the Gibbs ensemble describes thermal equilibrium (at least for "large systems") when the above physical identifications have been made is deep and incompletely clarified. (David Ruelle [91])

The question of using infinite systems as idealizations for studying the behavior of the finite ones is a hard mathematical problem. In this context, one of the most important approaches consists in replacing the original model by a mathematically tractable one taking care of that both systems result thermodynamically equivalent, in some proper sense. Anyway, the derivation of exact expressions for limit pressures can be avoided, in the case of very complicated models, by deriving upper and lower bounds for suitable correlation functions. Finally, the energy operators of a wide class of mean field models are diagonal with respect to the so-called number operators and in this scenario the associated limit thermodynamic functions can be derived by applying probabilistic methods. However these models depend on strong physical assumptions which reduces their range of applicability. In the next sections a brief account of that strategies will be given.

Although all these methods could give us a reasonable picture about how phase transitions really work, the idea of spontaneous breakdown of continuous symmetries in the thermodynamic limit as representative phenomenom of the changes of symmetries in real situations, keeps being, definitively, hard to grasp.

6. Emergentism vs reductionism?

I mention in this section the alleged dispute between emergentism and reductionism in science (particularly in physics and chemistry) just anecdotally, because it sounds to me a false conflict. The following paragraph, due to R. Laughin, clarifies the situation very well:

> From the reductionist standpoint, physical law is the motivating impulse of the universe. It does not come from anywhere and implies everything. From the emergentist perspective, physical law is a rule of collective behavior, it is a consequence of more primitive rules of behavior underneath (although it need not have been), and it gives one predictive power over a limited range of circumstances. Outside this range, it becomes irrelevant, supplanted by other rules that are either its children or its parents in a hierarchy of descent. Neither of these viewpoints can gain ascendancy over the other by means of facts, for both are fact-based and both are true in the traditional scientific sense of the term.The issue is more subtle a matter of institutional judgment. To paraphrase George Orwell, all facts are equal, but some are more equal than others [88].

7. Bose Einstein condensation

In this chapter, it will be shown that the usual theory of BEC, based on notions such as thermodynamic limit, is not necessarily contradictory with the phenomena described by experiments carried out in magnetic and optical traps for finite atom systems (mesoscopic systems), even taking into account that they are quite different in many aspects from the so-called textbook Bose- Einstein Condensation.

For someone equipped with a sharp vision or with very specialized sensory organs capable of detecting not only molecules or atoms but also the amount of them, conceptions such as infinite number of particles, could be very confusing. Moreover, this singular being might be unable to distinguish macroscopic objects as we do. Thus, it seems reasonable to think that a species accustomed to interaction with large but finite particle ensembles, at microscopic, mesoscopic and macroscopic levels, ought to develop a type of ad-hoc discrete mathematics to describe such a world where infinite systems have no sense. The question arises immediately: theoretically and experimentally speaking, are there firsthand phenomena in our physical reality to which we can access not only at microscopical level but also at a macroscopic level, in a similar way as the just described, enriching our own vision of the macroscopic world?

The phenomenon of Bose Einstein Condensation (BEC), predicted by Einstein in 1925 [10], corresponds to a macroscopic occupation of a single quantum state (ground state) by a large number of identical bosons (particles whose states are represented by symmetric wave functions).

Until recently the best experimental evidence that BEC could occur in a real physical system was the phenomenon of superfluidity in liquid helium as suggested originally by London[11] who introduced the concept of macroscopic occupation of the ground state and conjectured that the momentum-space condensation of bosons is enhanced by spatial repulsion between the particles [12]. However, nowadays, there exists a considerable amount of experimental evidence for BEC [13, 14].

Given the difficulty of the problem of proving the existence of BEC from a mathematical point of view, it is desirable to have idealized models in which one can develop concrete scenarios for BEC. In this sense, great efforts have been made to study Bose systems whose energy operators consider repulsive mean interactions represented by diagonal operators in the occupation numbers. It frequently leads to thermodynamically stable systems which can be classically understood.

BEC has been extensively studied, in the framework of quantum equilibrium statistical mechanics, as a kind of second order phase transition.

In this scenario, the theory predicts that at low temperatures and large densities of particles, quantum effects should become essential for the macroscopic behavior of the system. Moreover, under suitable assumptions, for some kind of models (homogeneous non interacting and weakly interacting Bose systems) displaying BEC, the mathematical formalism shows that a spontaneous symmetry breaking associated to gauge transformations may occur.

The development of highly sophisticated cooling techniques (laser cooling, vaporization) led to confirm, experimentally, in the case of diluted atomic gases trapped in magnetic or optical traps, the Einstein′s conjecture after 70 years (1995).

A significant experimental support was giving to the microscopic theory of superfluidity, closely related to BEC, by the observation in laboratory of the spectrum of elementary excitations [37] predicted by N.N. Bogoliubov in 1947 in an outstanding article [38].

Unlike all the previously mentioned theoretical many particle models, trapped gases are inhomogeneous and finite-sized systems. Even more, they can display low dimensional BEC, phenomenon prohibited for infinite Bose particle systems (Hohenberg theorem). Indeed, the number of atoms that can be put into the traps is not truly macroscopic. So far experiments have been carried out with a maximum of about 10^7 atoms. As a consequence, the thermodynamic limit is never reached exactly.

Thus, strictly speaking, in the context of this theory, such a behavior of trapped atoms is not a phase transition [92]. Moreover, W. Ketterle and van Drutten proved that the results obtained in finite size systems, in cases of certain dilute atomic gases, for some values of the critical parameters, such as chemical potential, temperature and condensate density, differs with those obtained in the thermodynamic limit. They have shown that the occupation of states of low energies, for these parameters, in that limit, disappear [9,10].

> [...] the existence of a mathematically sharp phase transition is not crucial to the description of real systems. What is important is the appearance of a 'condensation,' by which we mean the rapid accumulation of a substantial fraction of the N particles into the ground state (without big fluctuations about this average) when the temperature falls below a certain finite value. [93]

On the other hand, fragmentation of condensate, a novel phenomenom permitted at mesoscopic level, vanishes in the thermodynamic limit [98]. However, it is necessary to point out that these results were not at all surprising given the nature of the experiments. The quite obvious difference between the Bose gases confined in magnetic traps and the standard theory of BEC made evident the need to adapt the former mathematical strategies to the context of the observed phenomena or simply to develop them further. In other words, in the light of new discoveries, physicists and mathematicians are always compelled to reformulate or broaden old scientific definitions.

A relevant question is: why the just described phenomena are so relevant today? The answer is very simple: these systems, constituted by finite number of atoms, respond quite well to experimental manipulations at a fundamental level. Moreover, mesoscopic systems can be considered as a kind of bridge between the microscopic world and its macroscopic counterpart. In this sense, they represent the best theoretical and experimental scenario to analize a kind of real quantum many particle systems not only from a microscopic point of view but also from the perspective of the canonical ensemble.

On the other hand, there is little chance of finding physical phenomena that can be, simultaneously, both theoretically and experimentally studied firsthand as the case of BEC. It should contribute in the near future to a deeper comprehension of general phase transitions.

In this sense, a microscopical approach based on a constraint-cutoff mechanism of critical fluctuations (see [26] and references therein) has been used to study BEC in ultra-cold trapped gases, avoiding the infrared divergences. This approach will be discussed in a future essay.

8. Mathematical framework: Textbook BEC

In quantum mechanics, physical observables are given by an algebra of selfadjoint operators (generally unbounded) $(\mathcal{A}, *)$, where $*$ is an involution, defined on a suitable space of states (Hilbert space).

For a quantum system of many non interacting particles and in the framework of the first quantization formalism, the operator $S^l = -\frac{\triangle}{2}$, ($\triangle-$ laplacian) defined on the Hilbert space $\mathcal{H}^l = L^2(\Lambda_l)$, being $\Lambda_l = \left[-\frac{l}{2}, \frac{l}{2}\right]^\nu \subset \mathbb{R}^\nu$ a cubic box of boundary $\partial\Lambda_l$ and volume $V_l = l^\nu$, represents the one-particle energy operator.

Under suitable boundary conditions S^l becomes a self-adjoint operator on a dense set $\mathcal{D} \subset \mathcal{H}^l$, such that $S^l\varphi_{\mathbf{j}} = -\frac{1}{2}\triangle\varphi_{\mathbf{j}} = \lambda_l(\mathbf{j})\varphi_{\mathbf{j}}$ with $\mathbf{j} \in \mathbb{N} \cup \{0\}$, being $\{\varphi_{\mathbf{j}}\}$ a set of orthonormal eigenfunctions and $\lambda_l(0) < \lambda_l(1) \leq \lambda_l(2) \leq ....$ That is, for the particle system there exist both a countable set of accesible states as a countable set of accessible energies.

The second quantization formalism for many particle Bose systems consists in: 1. defining a Hilbert space for a system consisting of exactly N particles as $\mathcal{H}_{\mathrm{B}}^N := S_B\left(\otimes_{\mathbf{i}=1}^N \mathcal{H}_{\mathbf{i}}\right)$, where S_B is the so-called symmetrization operator, and $\mathcal{H}_{\mathbf{i}} = \mathcal{H}^l$ for all $\mathbf{i}$; 2. defining a Hibert space for a system with arbitrary number of particles, as $\mathcal{F}_{\mathrm{B}}(\mathcal{H}) := \oplus_{N=0}^\infty \mathcal{H}_{\mathrm{B}}^N$, (Fock space, with $\mathcal{H}_{\mathrm{B}}^0 = \mathbb{C}$; 3. introducing the operators of creation and annihilation of particles, $\hat{a}_{\mathbf{i}}, \hat{a}_{\mathbf{i}}^\dagger$, satisfying the commutation rules: $[\hat{a}_{\mathbf{i}}, \hat{a}_{\mathbf{j}}^\dagger] = \hat{a}_{\mathbf{i}}\hat{a}_{\mathbf{j}}^\dagger - \hat{a}_{\mathbf{j}}^\dagger\hat{a}_{\mathbf{i}} = \delta_{\mathbf{i},\mathbf{j}}$, where $\mathbf{j} \in \mathbb{N} \cup \{0\}$; 4. writing all observable A in terms of such operators, i.e., applying the rule $A \to A(\hat{a}, \hat{a}^\dagger)$.

If a selfadjoint operator $\hat{H}_l$ on the Hilbert space $\mathcal{F}_{\mathrm{B}}$, represents the energy operator of the system, the operator $\hat{H}_l^{(N)} = \hat{H}_l|_{\mathcal{H}_{\mathrm{B}}^N}$ is its restriction to the a space of exactly $N-$ particles. $\beta = T^{-1}$ and $\mu \in \mathbb{R}$ are the inverse temperature and the so-called chemical potential, respectively.

$\hat{H}_l(\mu)$ is defined as $\hat{H}_l(\mu) := \hat{H}_l - \mu\hat{N}$ where $\hat{N}$ is the total number operator given by $\hat{N} = \sum_{\mathbf{j}} \hat{n}_{\mathbf{j}}$ being $\hat{n}_{\mathbf{j}} = \hat{a}_{\mathbf{j}}^{\dagger}\hat{a}_{\mathbf{j}}$ the number operator associated to the $\mathbf{j}-$ mode.

$\hat{H}_l$ can be decomposed as the following sum: $\hat{H}_l = \hat{H}_l^0 + \hat{H}_l^I$, where $\hat{H}_l^0 = \sum_{\mathbf{j}} \lambda_l(\mathbf{j})\hat{a}_{\mathbf{j}}^{\dagger}\hat{a}_{\mathbf{j}}$ and $\hat{H}_l^I = \sum_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} V_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}}\hat{a}_{\mathbf{i}}^{\dagger}\hat{a}_{\mathbf{j}}^{\dagger}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{l}}$ are the seccond quantizations of the laplacian and of the interaction V, respectively.

With these definitions, at finite volume, it is possible to introduce the grand canonical partition function $\Xi_l(\beta,\mu)$ and the pressure $p_l(\beta,\mu)$:

$$\Xi_l(\beta,\mu) := \mathrm{Tr}_{\mathcal{F}_B} \exp\left(-\beta\hat{H}_l(\mu)\right),\ p_l(\beta,\mu) := \frac{1}{\beta V_l}\ln\Xi_l(\beta,\mu);$$

the canonical partition function $Z_{N,l}(\beta,\varrho)$ and the free energy $f_l(\beta,\varrho_l)$, where $\varrho_l = \frac{N}{V_l}$:

$$Z_{N,l}(\beta,\varrho) := \mathrm{Tr}_{\mathcal{H}_B^{(N)}} e^{-\beta\hat{H}_l^{(N)}},\quad f_l(\beta,\rho_l) := -\frac{1}{\beta V_l}\ln Z_{N,l}(\beta,\varrho),$$

and, finally, the Gibbs states in the grand canonical ensemble and in the canonical ensemble:

$$\langle\cdot\rangle_{\hat{H}_l(\mu)} = \Xi^{-1}(\beta,\mu)\,\mathrm{Tr}_{\mathcal{F}_B}\cdot\exp\left(-\beta\hat{H}_l(\mu)\right),$$

$$\langle \cdot \rangle_{\hat{H}_l^{(N)}} = Z_{N,l}(\beta, \varrho)\, \mathrm{Tr}_{\mathcal{H}_B^{(N)}} \cdot \exp\left(-\beta \hat{H}_l^{(N)}\right),$$

respectively.

The limit free energy $f(\beta, \varrho)$ and the limit pressure $p(\beta, \mu)$ are defined as:

$$f(\beta, \varrho) := \lim_{N_l, V_l \to \infty} f_l(\beta, \varrho_l), \text{ assuming that } \lim_{N_l, V_l \to \infty} \varrho_l = \varrho,$$

and

$$p(\beta, \mu) := \lim_{V_l \to \infty} p_l(\beta, \mu), \quad \text{assuming that}$$
$$\lim_{V_l \to \infty} \left\langle \frac{\hat{N}}{V_l} \right\rangle_{\hat{H}_l(\mu)} = \lim_{V_l \to \infty} \rho_l = \rho(\mu)$$

On the other hand, stable systems are defined as those for which there exists $\mu_* \in \mathbb{R}$ such that only for $\mu \in (-\infty, \mu_*]$, $p(\beta, \mu) < \infty$, while superstable systems satisfies $p(\beta, \mu) < \infty$. for all values of μ. Finally, if the following inequality (in the sense of operators)

$$\hat{H}_l^I \geq -\frac{C_2}{V_l}\hat{N} + \frac{C_1}{V_l}\hat{N}^2$$

holds, the system is superstable.

9. Types of BEC

- Condensation of type I corresponds to a macroscopic occupation of a finite nuber of states. Thus, a macroscopic occupation of the the ground state, or traditional Bose-Einstein condensation, is given by the fulfillment of the condition
$$\lim_{V_l\to\infty}\left\langle \frac{\hat{n}_{\mathbf{0}}}{V_l}\right\rangle_{\hat{H}_l(\mu)} = \rho_{\mathbf{0}} > 0.$$
For the latter, in the condensed-uncondensed phase transition the suitable order parameter is $\rho_{\mathbf{0}} = \rho - \rho_c$, being ρ_c a critical density.
- Condensation of type II holds when there exists an infinite number of states macroscopically occupied
- Condensation of the type III holds when there are not macroscopically occupied states but the following condition holds:

$$\lim_{\delta\to 0^+}\lim_{V_l\to\infty}\frac{1}{V_l}\sum_{\mathbf{p}\in\Lambda^*,\lambda_l(\mathbf{p})<\delta}\langle \hat{n}_{\mathbf{p}}\rangle_{\hat{H}_l(\mu)} > 0.$$

The third type of Bose condensation, denominated *generalized BEC* (GBEC), was introduced by M. Girardeau in 1960 [70].

GBEC is more robust that the other kinds of condensation in the sense that it is independent on the shape of the confining region. Indeed, in the case of the free Bose gas, it always occurs for particle density values larger than a critical one. Moreover, it has been demonstrated that this type of condensation is stable with respect to mean field perturbations of the free Bose gas [21]. In this framework a general theory of BEC was developed in ref.[73] for a noninteracting system of Bose particles.

10. Other BEC definitions

10.1. **Penrose-Onsager criterion.** Let $\Psi_{\mathbf{0}}^N(\mathbf{x}_1, \cdots, \mathbf{x}_N)$ be the normalized ground state in $\mathcal{H}_B^{(N)}$.

$$\gamma_{\mathbf{0}}(\mathbf{x}, \mathbf{x}') = N \int_{\mathbb{R}^{3(N-1)}} \Psi_{\mathbf{0}}^N(\mathbf{x}, \mathbf{x}_2, \cdot\cdot, \mathbf{x}_N) \overline{\Psi_{\mathbf{0}}^N(\mathbf{x}', \mathbf{x}_2, \cdot\cdot, \mathbf{x}_N)} \prod_{\mathbf{j}=2}^{N} d\mathbf{x_j}$$

$$\varrho_{\mathbf{0}}(\mathbf{x}) = \gamma_{\mathbf{0}}(\mathbf{x}, \mathbf{x}) = N \int_{\mathbb{R}^{3(N-1)}} |\Psi_{\mathbf{0}}^N(\mathbf{x}, \mathbf{x}_2, \cdot\cdot, \mathbf{x}_N)|^2 \prod_{\mathbf{j}=2}^{N} d\mathbf{x_j}.$$

In this case,

$$\mathrm{Tr}[\hat{\gamma}_{\mathbf{0}}] = \int \varrho_{\mathbf{0}}(\mathbf{x}) d\mathbf{x} = N.$$

The condition for Bose-Einstein condensation, in this framework, is:

$$\frac{1}{V} \int \int \gamma_{\mathbf{0}}(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = O(N),$$

if $\frac{N}{V} \to$ *cte.*, when $N, V \to \infty$.

10.2. **Diagonal and off diagonal long range orders.** The kernel of operator $\hat{\gamma}_{\mathbf{0}}$ is,

$$\gamma_{\mathbf{0}}(\mathbf{x}, \mathbf{x}') = n_{\mathbf{0}} \varphi_{\mathbf{0}}(\mathbf{x}) \bar{\varphi}_{\mathbf{0}}(\mathbf{x}') + \sum_{\mathbf{j}} n_{\mathbf{j}} \varphi_{\mathbf{j}}(\mathbf{x}) \bar{\varphi}_{\mathbf{j}}(\mathbf{x}')$$

For large N the sum can be replaced by an integral, vanishing when $||\mathbf{x}' - \mathbf{x}|| \to \infty$.

11. Ideal Bose gas

The Hamiltonian of the ideal Bose gas is given as:

$$\hat{H}_l^0 = \sum_{\mathbf{p}\in\Lambda_l^*} \lambda_l(\mathbf{p})\hat{n}_{\mathbf{p}}.$$

The sum runs over the set $\Lambda_l^* = \{\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3 : p_\alpha = 2\pi n_\alpha/l, n_\alpha \in \mathbb{Z}, \alpha = 1, 2, 3\}$. $\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{p}}$ are the Bose operators of creation and annihilation of particles defined on the Fock space $\mathcal{F}_B$, satisfying the usual commutation rules $[\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{p}}^\dagger] = \hat{a}_{\mathbf{q}}\hat{a}_{\mathbf{p}}^\dagger - \hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{q}} = \delta_{\mathbf{p},\mathbf{q}}I$, $\hat{n}_{\mathbf{p}} = \hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{p}}$ is the number operator associated to the mode $\mathbf{p}$. $\hat{N} = \sum\limits_{\mathbf{p}\in\Lambda_l^*} \hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{p}}$ is the total number operator and $\lambda_l(\mathbf{p}) = \mathbf{p}^2/2$.

Note that the Fock space$\mathcal{F}_B$ is isomorphic to the tensor product $\otimes_{\mathbf{p}\in\Lambda_l^*}\mathcal{F}_{\mathbf{p}}^B$ where $\mathcal{F}_{\mathbf{p}}^B$ is the one-mode Fock space constructed from the one-dimensional Hilbert space $\mathcal{H}_{\mathbf{p}} = \{\gamma e^{i\mathbf{p}\cdot\mathbf{x}}\}_{\gamma\in\mathbb{C}}$.

11.1. **Grand canonical thermodynamic sum.** For the ideal Bose gas, being $\mu < \lambda_l(\mathbf{0})$, we have,

$$\mathrm{Tr}_{\mathcal{F}_B(\mathcal{H}^l)}[e^{-\beta\hat{H}_l^0(\mu)}] = \prod_{\mathbf{p}\in\Lambda_l^*}\sum_{\{n_{\mathbf{p}}\}} e^{-\beta(\lambda_l(\mathbf{p})-\mu)n_{\mathbf{p}}} = \prod_{\mathbf{p}}(1-e^{-\beta(\lambda_l(\mathbf{p})-\mu)})^{-1}.$$

From this, the following expression for the finite pressure is obtained:

$$p_l^{(id)}(\beta,\mu) = -\frac{1}{\beta V_l}\sum_{\mathbf{p}\in\Lambda_l^*}\ln(1 - e^{-\beta(\lambda_l(\mathbf{p})-\mu)}).$$

This result leads to,

$$\langle \hat{n}_{\mathbf{p}} \rangle_{\hat{H}_l^0(\mu)} = \frac{1}{e^{\beta(\lambda_l(\mathbf{p})-\mu)} - 1}.$$

Finally, the following relation can be derived,

$$\rho_l^{(id)}(\mu) = \rho_{\mathbf{0},l}^{(id)}(\beta,\mu) + \frac{1}{V_l} \sum_{\mathbf{p}\in\Lambda_l^*\setminus\{0\}} \langle \hat{n}_j \rangle_{\hat{H}_l^0(\mu)},$$

where $\rho_l^{(id)}(\mu)$, $\rho_{\mathbf{0},l}^{(id)}(\beta,\mu)$ represent the total density of particles and the density of particles associated to the zero, with chemical potential α and finite volume V_l respectively.

Taking a sequence $\{\mu_l\}$ such that $\mu_l \to 0^-$, in the thermodynamic limit $V_l \to \infty$) it is obtained,

$$\rho^{(id)}(0) = \rho_{\mathbf{0}}^{(id)}(\beta,0) + \rho_c^{(id)}(\beta),$$

where it has been assumed that

$$\rho_c^{(id)}(\beta) = \lim_{V_l\to\infty} \frac{1}{V_l} \sum_{\mathbf{p}\in\Lambda_l^*\setminus\{0\}} \frac{1}{e^{\beta\lambda_l(\mathbf{p})} - 1} < \infty.$$

Therefore,

$$\rho_{\mathbf{0}}^{(id)}(\beta,0) = \rho^{(id)}(0) - \rho_c^{(id)}(\beta).$$

$\rho^{(id)}(0)$ is assumed constant while $\rho_c^{(id)}(\beta)$ is a decreasing function of β.. Thus, the condition for condensation is satisfied for values of β such that $\beta > \beta_c$, where β_c is the unique solution of the equation $\rho^{(id)}(0) = \rho_c^{(id)}(\beta)$. Note that Bose condensation in a 2-dimensional box can take place only at zero temperature.

On the other hand, assuming that $\hbar = 1,\ m = 1$ (m is the particle mass) and $\lambda = \sqrt{2\pi\beta}$, the density of particles at finite volume for the $d-$ dimensional free Bose gas is given as,

$$\rho_l = \frac{1}{V_l}\left(\frac{z}{1-z}\right) + \frac{1}{\lambda^d} g_{d/2}(z) = \rho_{\mathbf{0},l} + \frac{1}{\lambda^d} g_{d/2}(z)$$

being

$$g_{d/2}(z) = \frac{1}{\Gamma(d/2)} \int_0^\infty dx \frac{ze^{-x}x^{d/2-1}}{1-ze^{-x}}$$

$z = e^{\beta\mu}$ the fugacity and $\Gamma(c)$, $c \in \mathbb{C}\backslash\mathbb{R}^-$, the Gamma function.

For $d = 2$ we get,

$$\begin{aligned} g_1(z) = \frac{1}{\Gamma(1)} \int_0^\infty dx \frac{ze^{-x}}{1-ze^{-x}} &= \ln\left(1 - ze^{-x}\right)|_0^\infty \\ &= -\ln\left(1-z\right). \end{aligned}$$

Therefore, after passing to the thermodynamic limit, taking $\mu \to 0^-$, ρ goes to infinity. In conclusion, in two dimensions Bose-Einstein condensation is excluded.

A similar argument leads to the proof of absence of BEC in the case of the one-dimensional uniform ideal Bose gas.

On the other hand, it is possible to prove that BEC always holds for dimensions $d \geq 3$. Indeed, for $d = 3$, we have,

$$\rho_l = \frac{1}{V_l}\left(\frac{z}{1-z}\right) + \frac{1}{\lambda^3} g_{3/2}(z).$$

In this case, $g_{3/2}(z)$ is a bounded, positive, monotonically increasing function of $z \in [0, 1]$. Hence, since $g_{3/2}(1) = \zeta(3/2)$, where $\zeta(s)$ is the Riemann zeta function of s, $g_{3/2}(z) \leq \zeta(3/2)$ for all $z \in [0, 1]$.

Using these facts, in the thermodynamic limit we get,

$$\lambda^3\rho_{\mathbf{0}} = \lambda^3\rho - g_{3/2}(z) > \lambda^3\rho - g_{3/2}(1).$$

Therefore, $\rho_{\mathbf{0}} > 0$ when $\lambda^3\rho > g_{3/2}(1)$, i.e., under this condition the system undergoes BEC. Clearly, $\rho_c = \frac{1}{\lambda^3} g_{3/2}(1)$.

Note that the density of states for a homogeneous d-dimensional system is $\rho(\epsilon) = K_\gamma \epsilon^{\gamma-1}$, where K_γ is a constant and $\gamma = \frac{d}{2}$. However, for the harmonic oscillator $\gamma = d$.

12. Criticisms

> Conceptually, the necessity of the thermodynamic limit is an objectionable feature: first, the number of degrees of freedom in real systems, although possibly large, is finite, and, second, for systems with long-range interactions, the thermodynamic limit may even be not well defined. These observations indicate that the theoretical description of phase transitions, although very successful in certain aspects, may not be completely satisfactory [33].

It has been pointed out that the grand canonical ensemble misrepresents some physical quantities in presence of condensate, for example, by overestimating fluctuations of the condensate number (see for example [93] and references therein). To overcome this problem, the canonical ensemble has been extensively used, specifically on the basis of recurrence relations for $Z_{N,l}$. Moreover, differences between results obtained in the critical region of Bose -Einstein condensation by using the GCE statistics and others obtained in the framework of the canonical ensemble have been recently communicated:

> [...] the exact GCE result differs from the corresponding canonical-ensemble result by a factor on the order of unity even in the thermodynamic limit. Thus, a widely used GCE approach does not describe correctly the critical phenomena at the phase transition for the actual systems with a fixed number of particles and yields only an asymptotic far outside the critical region.

13. Conservation laws and selection rules

In 1918, Emmy Noether, established the connection between continuous symmetries and conservation laws in nature. For example, translational invariance to conservation of the lineal momentum, time invariance leads to energy conservation, etc.

If

$$[O, \hat{H}_l(\mu)] = 0,$$

being

$$O(t) = e^{-it\hat{H}_l(\mu)} O e^{it\hat{H}_l(\mu)},$$

it follows that

$$\frac{dO(t)}{dt} = -\frac{i}{\hbar}[O(t), \hat{H}_l(\mu)] = 0.$$

If $O = \hat{N}$ satisfies the above commutation rule, the particles number is conserved in the system. In this case, $\hat{H}_l(\mu)$ becomes invariant under the gauge transformations associated to the group $U(1)$ given as

$$\hat{a}_{\mathbf{p}} \to e^{i\varphi}\hat{a}_{\mathbf{p}}, \;\; \hat{a}^{\dagger}_{\mathbf{p}} \to e^{-i\varphi}\hat{a}^{\dagger}_{\mathbf{p}}.$$

From this, the following selection rules

$$\left\langle \hat{a}^{\dagger}_{\mathbf{p}_1} \cdot \cdot \hat{a}^{\dagger}_{\mathbf{p}_r} \hat{a}_{\mathbf{q}_1} \cdot \cdot \hat{a}_{\mathbf{q}_s} \right\rangle_{\hat{H}_l(\mu)} = 0, \text{ if } r \neq s,$$

hold.

In particular, $\left\langle \hat{a}^{\dagger}_{\mathbf{p}} \right\rangle_{\hat{H}_l(\mu)} = \left\langle \hat{a}_{\mathbf{p}} \right\rangle_{\hat{H}_l(\mu)} = V_l \eta_l = 0.$

On the other hand, translational invariance is associated to the conservation of the total momentum, whose associated operator is given by,

$$\mathbf{P} = \sum_{\mathbf{p} \in \Lambda_l} \mathbf{p} \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}}$$

In this case,

$$[\hat{H}_l(\mu), \mathbf{P}] = 0 \Rightarrow \left\langle \hat{a}^\dagger_{\mathbf{p}_1} \cdot \cdot \hat{a}^\dagger_{\mathbf{p}_r} \hat{a}_{\mathbf{q}_1} \cdot \cdot \hat{a}_{\mathbf{q}_s} \right\rangle_{\hat{H}_l(\mu)} = 0,$$

$$\text{if } \ \mathbf{p}_1 + ... + \mathbf{p}_r \neq \mathbf{q}_1 + ... + \mathbf{p}_s.$$

$\hat{N}$, and $\mathbf{P}$ are denominated “symmetry generators”.

14. Symmetry breaking and groundstate

The study of the so-called spontaneous symmetry breaking of continuous symmetry, fundamental notion in quantum statistical physics, has received important contributions of many scientists, among them,Y. Nambu [134], J. Goldstone [99-100], P. Higgs [135], S. Weinberg [136] and N.N. Bogoliubov [91]. The latter one introduced conceptions such as the coherent macroscopic state with a fixed phase and the displacement canonical transformation of the field operator describing the Bose-Einstein condensate in the framework of the study of the free Bose gas and in the case of a superfluid model.

A crucial role in the standard theory of quantum phase transitions has been historically played by the notion of order parameter, defined as the thermal average -of certain operator-presenting one or more gaps in the thermodynamic limit as function depending on the inverse temperature and the chemical potential (grand canonical ensemble) or on the density of particles (canonical ensemble).

At finite volume, a continuous symmetry is associated with many infinitely degenerated ground states connected between them by unitary symmetry transformations. In this sense, these states are physically equivalent, having the same energy, and being the ground state of the system, sometimes, understood as a superposition of them. However, in the thermodynamic limit, these connections vanish, and an infinite number of inequivalent ground states arise.

On the other hand it is difficult to speak about exact symmetries if their are being permanently broken by small disturbances. Mathematically speaking, the disturbance once chosen and provided that the parameters on which it depends are fixed, selects a unique ground state for the system. This fact leads, in a natural way, to obtain analytical expressions for the order parameter, which can take, after passing to the thermodynamic limit and switching off perturbation, different values, depending on the choice of the mentioned vacuum.

Finally, one of the most significant consequences of the spontaneous breaking of a symmetry is the emergency of soft modes in the energy spectrum, the so-called Nambu-Goldstone modes.

> [...] NG quanta are not simply mathematical constructs. They are realistic physical boson particles, dynamically generated by SSB. They undergo scattering with other particles of the system or with observational probes, as for example in neutron-phonon scattering in crystals).[176]

15. BEC and spontaneous symmetry breaking (SSB)

The standard strategy devoted to eliminate the before mentioned degeneracy consists in introducing a small term on the original energy operator, preserving its self-adjointness but eliminating the symmetry corresponding to some conservation law.

Thus, it is possible to break the global $U(1)$ symmetry by adding the extra term $-\sqrt{V_l}h\left(\hat{a}_{\mathbf{0}}e^{-i\varphi}+\hat{a}_{\mathbf{0}}^{\dagger}e^{i\varphi}\right)$ to the original energy operator obtaining the new Hamiltonian $\hat{H}_{l,h,\phi}(\mu)=\hat{H}_l(\mu)-\sqrt{V_l}h\left(\hat{a}_{\mathbf{0}}e^{-i\varphi}+\hat{a}_{\mathbf{0}}^{\dagger}e^{i\varphi}\right)$ for which $[\hat{N},\hat{H}_{l,h,\varphi}(\mu)]\neq 0$, being $h\in\mathbb{R}^+,\ \varphi\in[0,2\pi)$. In this case, the former selection rules, at finite volume, do not hold anymore, i.e.:

$$|\left\langle \hat{a}_{\mathbf{p}}^{\dagger}\right\rangle_{\hat{H}_{l,h,\varphi}(\mu)}|=|\left\langle \hat{a}_{\mathbf{p}}\right\rangle_{\hat{H}_{l,h,\varphi}(\mu)}|=\sqrt{V_l}\eta_{l,h,\varphi}\neq 0.$$

For a Bose system undergoing BEC, in the thermodynamic limit, we have

$$\lim_{h\to 0}\lim_{V_l\to\infty}\eta_l^2=\begin{cases}\rho_{\mathbf{0}}\neq 0 & \text{if } \rho>\rho_c\\ 0 & \text{if } \rho\leq\rho_c\end{cases}$$

From a mathematical point of view, for $\rho\leq\rho_c$, in the uncondensed phase, it is possible to make a limit exchange, obtaining:

$$\lim_{h\to 0}\lim_{V_l\to\infty}\eta_{l,h,\varphi}=\lim_{V_l\to\infty}\lim_{h\to 0}\eta_{l,h,\varphi}=0.$$

However, for $\rho>\rho_c$,

$$\lim_{h\to 0}\lim_{V_l\to\infty}\eta_{l,h,\varphi}\neq\lim_{V_l\to\infty}\lim_{h\to 0}\eta_{l,h,\varphi}$$

In this context the limit thermal averages defined as

$$\prec \cdot \succ := \lim_{h\to 0} \lim_{V_l\to\infty} \langle - \rangle_{\hat{H}_{l,h,\varphi}(\mu)}$$

have been denominated *Bogoliubov quasiaverages* or *anomalous averages*.

Thus, the degeneracy of regular averages, produced by the presence of additive conservation laws (or equivalently, by the invariance of the Hamiltonian with respect to certain groups of transformations) is reflected by the dependence of quasi averages on the extra infinitesimal term. In this sense N. N. Bogoliubov claimed that the latter are more "physical" than the regular averages [91]. However this procedure, in some cases, has been applied *without having necessarily a clear physical meaning*.

> We are assuming that other types of degeneracy do not exist and, thus, the introduction of the term [...] is sufficient for the removal of the degeneracy. (N. N. Bogoliubov [91])

Let $\hat{\rho}_{\mathbf{0},l} = V_l^{-1}\hat{a}_{\mathbf{0}}^{\dagger}\hat{a}_{\mathbf{0}}$, $\hat{\eta}_l = V_l^{-\frac{1}{2}}\hat{a}_{\mathbf{0}}$. In the case of the free Bose gas, for

$$\hat{H}^0_{l,h,\varphi} = \sum_{\mathbf{p}} \lambda_l(\mathbf{p})\hat{a}_{\mathbf{p}}^{\dagger}\hat{a}_{\mathbf{p}} - \sqrt{V_l}h\left(\hat{a}_{\mathbf{0}}e^{-i\varphi} + \hat{a}_{\mathbf{0}}^{\dagger}e^{i\varphi}\right),$$

being $\Lambda_l^* = \{\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{R}^d : p_\alpha = 2\pi n_\alpha / l, n_\alpha \in \mathbb{Z}, \alpha = 1, 2, ..., d\}$, and $\lambda_l(\mathbf{p}) = \mathbf{p}^2/2$, the following limits

$$\lim_{h\to 0^+} \lim_{V_l\to\infty} \langle \hat{\rho}_{\mathbf{0},l} \rangle_{\hat{H}_{l,h,\varphi}(\mu)} = \rho_{\mathbf{0}}, \quad \lim_{h\to 0^+} \lim_{V_l\to\infty} \langle \hat{\eta}_l \rangle_{\hat{H}_{l,h,\varphi}(\mu)} = \sqrt{\rho_{\mathbf{0}}} e^{i\varphi}$$

hold [91]. In other words:

$$\lim_{h\to 0^+} \lim_{V_l\to\infty} \langle \hat{\rho}_{\mathbf{0},l} \rangle_{\hat{H}_{l,h,\varphi}(\mu)} = \lim_{h\to 0^+} \lim_{V_l\to\infty} | \langle \hat{\eta}_l \rangle_{\hat{H}_{l,h,\varphi}(\mu)} |^2 = \rho_{\mathbf{0}}.$$

Let us consider the following perturbation of the free Hamiltonian:

$$\hat{H}_{l,\nu}(\mu) = \hat{H}_l^0(\mu) - \sqrt{V}\nu(e^{i\theta}\hat{a}_0^\dagger + e^{-i\theta}\hat{a}_0).$$

Let

$$\hat{a}_{\mathbf{0}} = -\frac{\nu}{\mu} e^{i\theta} \sqrt{V} + \hat{b}_{\mathbf{0}}, \quad \hat{a}_{\mathbf{0}}^\dagger = -\frac{\nu}{\mu} e^{-i\theta} \sqrt{V} + \hat{b}_{\mathbf{0}}^\dagger$$

Substituting this operators in the original energy operator,

$$\hat{H}_l^0 = -\mu \hat{b}_{\mathbf{0}}^\dagger \hat{b}_{\mathbf{0}} + \sum_{\mathbf{p}\in\Lambda_l{}^*\setminus\{0\}} \left(\frac{\mathbf{p}^2}{2} - \mu\right) \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{\nu^2 V}{\mu}.$$

It is assumed that,

$$\mu = -\frac{\nu}{\sqrt{\rho_{0,l}}}.$$

Clearly,

$$\left\langle \hat{b}_{\mathbf{0}}^\dagger \right\rangle_{\hat{H}_{l,\nu}^0(\mu)} = \left\langle \hat{b}_{\mathbf{0}} \right\rangle_{\hat{H}_{l,\nu}^0(\mu)} = 0.$$

Besides,

$$\left\langle \hat{b}_{\mathbf{0}}^\dagger \hat{b}_{\mathbf{0}} \right\rangle_{\hat{H}_{l,\nu}^0(\mu)} = \left(\exp\beta\left(\frac{\nu}{\sqrt{\rho_{\mathbf{0}}}}\right) - 1\right)^{-1}$$

$$\langle \hat{n}_{\mathbf{p}} \rangle_{\hat{H}^0_{l,\nu}(\mu)} = \left(\exp \beta \left(\frac{\nu}{\sqrt{\rho_{\mathbf{0}}}} + \frac{\mathbf{p}^2}{2} \right) - 1 \right)^{-1}.$$

Then,

$$\rho_l = \rho_{\mathbf{0},l} + \frac{1}{V} \sum_{\mathbf{p} \in \Lambda_l{}^* \setminus \{0\}} \left(\exp \beta \left(\frac{\nu}{\sqrt{\rho_{\mathbf{0}}}} + \frac{\mathbf{p}^2}{2} \right) - 1 \right)^{-1}.$$

Passing to the thermodynamic limit,

$$\rho = \rho_{\mathbf{0}} + \frac{1}{(2\pi)^3} \int \left(\exp \beta \left(\frac{\nu}{\sqrt{\rho_{\mathbf{0}}}} + \frac{\mathbf{p}^2}{2} \right) - 1 \right)^{-1} d^3\mathbf{p}.$$

On the other hand,

$$\left\langle \frac{\hat{b}^\dagger_{\mathbf{0}} \hat{b}_{\mathbf{0}}}{V} \right\rangle_{\hat{H}^0_{l,\nu}(\mu)} = \left\langle \left(\frac{\hat{a}^\dagger_{\mathbf{0}}}{\sqrt{V}} - \sqrt{\rho_{\mathbf{0},l}} e^{-i\theta} \right) \left(\frac{\hat{a}_{\mathbf{0}}}{\sqrt{V}} - \sqrt{\rho_{\mathbf{0},l}} e^{i\theta} \right) \right\rangle_{\hat{H}^0_{l,\nu}(\mu)}$$

$$= \lim_{V \to \infty} \frac{1}{V} \left(\exp \beta \left(\frac{\nu}{\sqrt{\rho_{\mathbf{0},l}}} \right) - 1 \right)^{-1} = 0.$$

This leads to,

$$\lim_{V \to \infty} \frac{1}{V} (\left\langle \hat{a}^\dagger_{\mathbf{0}} \hat{a}_{\mathbf{0}} \right\rangle_{\hat{H}^0_{l,\nu}(\mu)} - \sqrt{\rho_{\mathbf{0},l}} e^{-i\theta} \left\langle \frac{\hat{a}^\dagger_{\mathbf{0}}}{\sqrt{V}} \right\rangle_{\hat{H}^0_{l,\nu}(\mu)}$$

$$- \sqrt{\rho_{\mathbf{0},l}} e^{i\theta} \left\langle \frac{\hat{a}_{\mathbf{0}}}{\sqrt{V}} \right\rangle_{\hat{H}^0_{l,\nu}(\mu)} + \rho_l) = 0.$$

In other words,

$$\rho_{\mathbf{0}} = \lim_{V\to\infty} \left| \left\langle \frac{\hat{a}_{\mathbf{0}}^{\dagger}}{\sqrt{V}} \right\rangle_{\hat{H}^0_{l,\nu}(\mu)} \right|^2$$

15.1. **Ground state.** Let, $\Psi_{1,\mathbf{0}} \neq 0$ be the vacuum of the original ideal system. At finite volume V, the transformation

$$\hat{b}_{\mathbf{0}} = U(z)\hat{a}_{\mathbf{0}}U^{-1}(z) = \hat{a}_{\mathbf{0}} - zI, \text{ where } U(z) = \exp\{\bar{z}\hat{a}_{\mathbf{0}} - z\hat{a}_{\mathbf{0}}^{\dagger}\},$$

and $z \in \mathbb{C}$, is a unitary equivalent representation of the Bose commutation rules.

Clearly, $\hat{b}_{\mathbf{0}}\Psi_{1,\mathbf{0}} = -z\Psi_{1,\mathbf{0}} \neq 0,$ for $z \neq 0$, therefore $\Psi_{1,\mathbf{0}}$ does not represent the vacuum of $\hat{b}_{\mathbf{0}}$.

On the other hand,

$$U(z)\hat{a}_{\mathbf{0}}\Psi_{1,\mathbf{0}} = \hat{b}_{\mathbf{0}}\left(U(z)\Psi_{1,\mathbf{0}}\right) = 0 \Rightarrow \Psi_{2,\mathbf{0}}(z) = U(z)\Psi_{1,\mathbf{0}}.$$

corresponds to the vacuum of $\hat{b}_{\mathbf{0}}$.

From the Baker-Campbell-Hausdorff formula it follows that,

$$U(z) = \exp\left\{-\frac{1}{2}|z|^2\right\}\exp\{-z\hat{a}_{\mathbf{0}}^{\dagger}\}\exp\{\bar{z}\hat{a}_{\mathbf{0}}\}$$

$$\begin{aligned}\Psi_{2,\mathbf{0}}(z) &= \exp\left\{-\frac{1}{2}|z|^2\right\}\exp\{-z\hat{a}_{\mathbf{0}}^{\dagger}\}\Psi_{1,\mathbf{0}} \\ &= \exp\left\{-\frac{1}{2}|z|^2\right\}\sum_{n=0}^{\infty}\frac{(-z)^n}{\sqrt{n!}}\Psi_{1,n}.\end{aligned}$$

This means that, the ground state $\Psi_{2,\mathbf{0}}(z)$ is a coherent state which can be written as a linear combination of Fock space functions. In other words, we have a set of equivalent Fock spaces. Moreover, in the case of the free Bose gas, the substitution $\hat{a}_{\mathbf{0}} = z\sqrt{V} + \hat{b}_{\mathbf{0}}$ leads to

$$\langle \Psi_{1,\mathbf{0}},\ \Psi_{2,\mathbf{0}} \rangle = \exp\left\{ -\frac{1}{2} V |z|^2 \right\} \to 0, \text{ when } \ V \to \infty.$$

Thus, in the infinite system $\Psi_{1,\mathbf{0}} \perp \Psi_{2,\mathbf{0}}$, i.e., we have unitary inequivalent representations for the Bose commutation rules.

By fixing the value of z as $\sqrt{\rho_{\mathbf{0}}} e^{i\varphi}$, the value of the order parameter has been also fixed.

Besides, note that

$$\langle \Psi_{2,\mathbf{0}}, \hat{a}_{\mathbf{0}} \Psi_{2,\mathbf{0}} \rangle = \left\langle \Psi_{2,\mathbf{0}}, \left(\sqrt{V\rho_{\mathbf{0}}} e^{i\varphi} + \hat{b}_{\mathbf{0}} \right) \Psi_{2,\mathbf{0}} \right\rangle = \sqrt{V\rho_{\mathbf{0}}} e^{i\varphi}.$$

Analogously,

$$\left\langle \Psi_{2,\mathbf{0}}, \hat{a}^{\dagger}_{\mathbf{0}} \Psi_{2,\mathbf{0}} \right\rangle = \sqrt{V\rho_{\mathbf{0}}} e^{-i\varphi}.$$

Thus, for an infinite system, we have: $\Psi_{1,\mathbf{0}} \perp \Psi_{2,\mathbf{0}}$, i.e., infinite states of minimum energy, all orthogonal to each other appear or, in other words, we have infinite inequivalent representations of the commutation rules.

On the other hand, by fixing z as $\sqrt{\rho_{\mathbf{0}}} e^{i\varphi}$, we also fix the value of the expression $\mathbb{E}_{\Psi_{2,\mathbf{0}}}(\hat{a}_{\mathbf{0}})$, where

$$\begin{aligned} \mathbb{E}_{\Psi_{2,\mathbf{0}}}(\hat{a}_{\mathbf{0}}) = \langle \Psi_{2,\mathbf{0}}, \hat{a}_{\mathbf{0}} \Psi_{2,\mathbf{0}} \rangle &= \left\langle \Psi_{2,\mathbf{0}}, \left(\sqrt{V\rho_{\mathbf{0}}} e^{i\varphi} + \hat{b}_{\mathbf{0}} \right) \Psi_{2,\mathbf{0}} \right\rangle \\ &= \sqrt{V\rho_{\mathbf{0}}} e^{i\varphi}. \end{aligned}$$

It is possible to prove that,

$$\mathbb{E}_{\Psi_{2,\mathbf{0}}}(\hat{a}_{\mathbf{0}}^{\dagger}) = \left\langle \Psi_{2,\mathbf{0}}, \hat{a}_{\mathbf{0}}^{\dagger}\Psi_{2,\mathbf{0}} \right\rangle = \sqrt{V\rho_{\mathbf{0}}}e^{-i\varphi},$$

i.e.:

$$\mathbb{E}_{\Psi_{2,\mathbf{0}}}\left(\frac{\hat{a}_{\mathbf{0}}}{V}\right) = \sqrt{\rho_{\mathbf{0}}}e^{i\varphi}, \;\; \mathbb{E}_{\Psi_{2,\mathbf{0}}}\left(\frac{\hat{a}_{\mathbf{0}}^{\dagger}}{V}\right) = \sqrt{\rho_{\mathbf{0}}}e^{-i\varphi}.$$

This values remain constant in the thermodynamic llimit

Therefore, if $\rho_{\mathbf{0}}$ represents the amount of condensate in the state $\Psi_{2,\mathbf{0}}$, orthogonal to $\Psi_{1,\mathbf{0}}$ in the limit. Moreover,

$$\mathbb{E}_{\Psi_{2,\mathbf{0}}}(\hat{a}_{\mathbf{0}}^{\dagger})\mathbb{E}_{\Psi_{2,\mathbf{0}}}(\hat{a}_{\mathbf{0}}) = \rho_{\mathbf{0}}.$$

This also follows for systems with Hamiltonians of the type [98]:

$$\hat{H}_{l,h,\delta} = \sum_{\mathbf{p}\in\Lambda_l{}^*} \lambda_l(\mathbf{p})\hat{a}_{\mathbf{p}}^{\dagger}\hat{a}_{\mathbf{p}} + \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}\in\Lambda_l^*} \hat{V}(\mathbf{q})\hat{a}_{\mathbf{p}+\mathbf{q}}^{\dagger}\hat{a}_{\mathbf{p}'-\mathbf{q}}^{\dagger}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{p}'} -$$

$$-\sqrt{V_l}h\left(\hat{a}_{\mathbf{0}}e^{-i\varphi} + \hat{a}_{\mathbf{0}}^{\dagger}e^{i\varphi}\right),$$

where $\hat{V}(\mathbf{q})$ is the Fourier transform of the two body potential $V(\mathbf{r})$, i.e.,

$$\hat{V}(\mathbf{q}) = \frac{1}{V}\sum_{\mathbf{x},\mathbf{y}\in\Lambda_l} V(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}, \;\; \mathbf{r} = \mathbf{x} - \mathbf{y},$$

satisfying $|\hat{V}(\mathbf{q})| \leq \varphi < \infty$. See also an alternative derivation of this result given by A. Sütő in [83].

> The non-commuting limits [...] show that any tiny perturbation is sufficient to break the symmetry. No real-world system is completely free of external perturbation. Thus the limiting procedure of maintaining a small external field while taking the thermodynamic limit could be viewed not only as a mathematical trick but also to good approximation as the actual situation in nature as well [153]

According to these arguments this procedure would reflect the instability of the thermal averages against to small perturbations involving the violation of the gauge symmetry. However, we must remember again that both Bose condensation as SSB occur only in the thermodynamic limit which in real physical systems is never reached. Moreover an underlying question is whether there is only one restricted class of perturbations compatible with the chosen order parameter, with the existence of pure states and ODLRO (off diagonal long range order).

At this point let us remember that generally speaking infinite equilibrium states should behave as ergodic states. Thus, unlike the original Gibbs state $\langle\cdot\rangle_{\hat{H}_{l,h\varphi}(\mu)}$, being A, B local observables written in terms of operators of creation and annihilation operators and $\tau_{\mathbf{x}}$ the map representing a translation in $\mathbf{x}\in\mathbb{R}^\nu$, $\mathbf{x}\in\mathbb{R}^\nu$, the Bogoliubov quasi-average $\prec\cdot\succ$ satisfies the ergodicity condition:

$$\lim_{||\mathbf{x}||\to\infty} \prec A\tau_{\mathbf{x}}(B)\succ = \prec A\succ\prec B\succ .$$

In this context the quasi-average is a more adequate notion, in a physical sense, than the degenerate equilibrium state for studying many particle Bose systems. Indeed, in a recent work [101] it has been shown that *Bogoliubov quasi-averages select the pure or ergodic states in the ergodic decomposition of the thermal (Gibbs) state.*

> [...] we reexamined the issue of ODLRO versus SSB by the method of Bogoliubov quasi-averages, commonly regarded as a symmetry-breaking trick. We showed that it represents a general method of construction of extremal, pure or ergodic states, both for quantum spin systems [...] and many-body Boson systems [...] (W. F. Wreszinski, V. A. Zagrebnov[101])

The presence of such particular kind of disturbances would be not only a sufficient condition but a necessary for the existence of ergodic states, identified by the standard theory of phase transitions as pure states (or pure phases). Thus, the above mentioned theoretical result tends rather to complicate the scenario than to clarify it in the sense that only a very particular type of perturbation would be capable of producing a symmetry rupture consistent with this theory.

A. Verbeure commented,

> We should note that SSB can occur only in the thermodynamic limit formulation of equilibrium [...] It is also clear that adding a symmetry breaking term to the Hamiltonian is a forced way of breaking the symmetry of the system and is not the same as the phenomenon of spontaneous breaking of the symmetry [102].

In this sense, the introduction of an external field does not explain by itself the broken of symmetry and therefore the notion of asymptotic approximation, in this case, should be clarified. However, for finite systems, and V_l large enough, the following behavior should be expected to occur:

$$\left\langle \frac{\hat{a}_{\mathbf{0}}^{\dagger}\hat{a}_{\mathbf{0}}}{V_l} \right\rangle_{\hat{H}_{l,h,\varphi}(\mu)} \sim \rho_{\mathbf{0}}, \quad \left\langle \frac{\hat{a}_{\mathbf{0}}}{\sqrt{V_l}} \right\rangle_{\hat{H}_{l,h,\varphi}(\mu)} \sim \sqrt{\rho_{\mathbf{0}}} e^{i\varphi}.$$

From this it can not be determined whether the transition, if it exists, is spontaneous or not, however, once the condensed and uncondensed phases emerge, the thermodynamic limit, as an approximation of the real phenomena far away from critical regions, seems to be a suitable scenario for determining numerical estimations for the condensate density and for the order parameter defining the breakdown of symmetry for finite but large enough systems. However, this is a conjecture that should be confirmed or rejected on the basis of experimental evidence.

On the other hand, the introduction of that kind of external fields implies strong technical constrains on the possible experiments to be carried out for verifying that asymptotic behaviour. In this sense, in absence of specially designed and controled laboratory experiences: what could be the probability of occurrence of such a macroscopic behaviour in the physical realm? Does spontaneous gauge symmetry breaking represent really a contingent phenomenon? Despite a physical model must satisfy conditions of plausibility and solubility [103], the scope of both notions is not clearly stated in this particular case. Moreover, in absence of macroscopic occupation of the ground state [104], there will be no spontaneous symmetry breaking, even in the presence of GBEC (or nonextensive BEC). In other words, the inclusion of these fields, by itself, does not guarantee the symmetry breaking in the thermodynamic limit. However, at mesoscopic level the absence of selection rules may appear as an apparent breakdown of the continuous symmetry although this effect is destined to disappear in the aforementioned limit.

Thus, the introduction of quasi averages seems to be motivated by a plausible physical consideration: *many real phase transitions are accompanied by the breakdown of symmetries.* A consequence of a global symmetry breakdown is the emergence of the so-called massless Goldstone modes [99,100]. Their appearance in experiments associated to Bose-Einstein condensation of magnons, in neighborhoods of critical regions, has been recently reported.

> The interplay between spontaneously broken gauge symmetries and Bose-Einstein condensation has long been controversially discussed in science, since the equation of motions are invariant under phase transformations. Within the present model it is illustrated that spontaneous symmetry breaking appears as a non-local process in position space, but within disjoint subspaces of the underlying Hilbert space. Numerical simulations show that it is the symmetry of the relative phase distribution between condensate and non-condensate quantum fields which is spontaneously broken when passing the critical temperature

> for Bose-Einstein condensation. Since the total number of gas particles remains constant over time, the global U(1)-gauge symmetry of the system is preserved.(Alexej Schelle [124])

Would it be a mistake to think that the breakdown of the spontaneous symmetry is a necessary and sufficient physical condition for the bosonic condensation? Is it true only if from the beginning we consider a model including a field of the above described type? Indeed, could it be ensured that a system free of these kind of small and special disturbances can condense?

> Fortunately, we can understand BEC and superfluidity without breaking the $U(1)$ symmetry, moreover, theories with and without the $U(1)$ gauge theory virtually yield the same results in the thermodynamic limit [154].

16. Weakly interacting Bose gas and superfluidity. Bogoliubov approach

The denominated weakly interacting Bose gas, mathematically determined by a contact interaction potential $V(\mathbf{x},\mathbf{y}) = g\delta(\mathbf{x},\mathbf{y})$ where g represents the interaction coupling strength in the second quantization formalism has a Hamiltonian given as

$$\hat{H}_l = \sum_{\mathbf{p}\in\Lambda_l{}^*} \frac{\mathbf{p}^2}{2}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{g}{2V_l}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}\in\Lambda_l^*}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}'}\hat{a}_{\mathbf{p}+\mathbf{q}}\hat{a}_{\mathbf{p}'-\mathbf{q}}.$$

In this case we have considered the particle mass m equal to one.

The fundamental idea of Bogoliubov consisted in obtaining a quadratic approximation of the original Hamiltonian by neglecting terms with three and four operators obtaining an energy operator which can be diagonalized by a using a suitable canonical transformation.

$$\hat{H}_l = \sum_{\mathbf{p}\in\Lambda_l{}^*} \frac{\mathbf{p}^2}{2}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} +$$

$$\frac{g}{2V_l}\left\{(\hat{a}^\dagger_{\mathbf{0}})^2\hat{a}^2_{\mathbf{0}} + \sum_{\mathbf{p}\in\Lambda_l^*\setminus\{\mathbf{0}\}}\left(2\hat{a}^\dagger_{\mathbf{0}}\hat{a}_{\mathbf{0}}\hat{n}_{\mathbf{p}} + \hat{a}^2_{\mathbf{0}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}} + (\hat{a}^\dagger_{\mathbf{0}})^2\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}}\right)\right\}.$$

Note that $\hat{H}_l$ still commutes with the total number operator $\hat{N} = \sum_{\mathbf{p}}\hat{n}_{\mathbf{p}}$.

In this approach it is assumed not only that the mean number of particles in the condensate is large enough (10^{23} for liquid ^{4}He) at very low temperatures but that almost all particles are in it. Moreover, it is conjectured that the creation and annihilation operators associated to zero mode, under such conditions, behave themselves as c-numbers, i.e., commute with each other and, for this reason, can be substituted by $\sqrt{VN}$ or $\sqrt{Vn_{\mathbf{0}}}$, where N and $n_{\mathbf{0}}$ represents the total number of particles and the number of particles in the ground state, respectively ($n_{\mathbf{0}} \approx N$).

$$\hat{H}_l' = \sum_{\mathbf{p}\in\Lambda_l{}^*} \frac{\mathbf{p}^2}{2}\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{gN}{V} \sum_{\mathbf{p}\in\Lambda_l{}^*\setminus\{\mathbf{0}\}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}$$
$$+\frac{gN}{2V} \sum_{\mathbf{p}\in\Lambda_l{}^*\setminus\{\mathbf{0}\}} \left(\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger + \hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}}\right) + \frac{gN^2}{2V}.$$

Clearly, $\hat{H}_l'$ is not invariant under the transformations associated to the group $U(1)$, before mentioned,

$$\hat{a}_{\mathbf{p}} \to e^{i\varphi}\hat{a}_{\mathbf{p}}, \;\; \hat{a}_{\mathbf{p}}^\dagger \to e^{-i\varphi}\hat{a}_{\mathbf{p}}^\dagger.$$

This means that the total number of particles in the system is not preserved. Fortunately, being $\hat{H}_l'$ a quadratic form in terms of the creation and annihilation operators, can be diagonalized by using the well-known transformations,

$$\hat{a}_{\mathbf{p}} = u_{\mathbf{p}}\hat{b}_{\mathbf{p}} - v_{\mathbf{p}}\hat{b}_{-\mathbf{p}}^\dagger, \;\; \hat{a}_{-\mathbf{p}}^\dagger = u_{\mathbf{p}}\hat{b}_{-\mathbf{p}}^\dagger - v_{\mathbf{p}}\hat{b}_{\mathbf{p}},$$

with

$$u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2 = 1$$

and by defining

$$\varrho = \frac{N}{V},\ \varrho_{\mathbf{0}} = \frac{n_{\mathbf{0}}}{V}.$$

Thus, in terms of these new operators, representing elementary excitations (**Bogolyubov excitations** or **quasiparticles**), $\hat{H}_l'$ takes the form,

$$\hat{H}_l'' = \sum_{\mathbf{p}\in{\Lambda_l}^*\setminus\{\mathbf{0}\}} \epsilon_l(\mathbf{p})\hat{b}_{\mathbf{p}}^{\dagger}\hat{b}_{\mathbf{p}} + 2\xi_l(\mathbf{p})v_{\mathbf{p}}^2 - g\varrho 2u_{\mathbf{p}}v_{\mathbf{p}}$$

where the elementary excitations spectrum is given as,

$$\epsilon_l(\mathbf{p}) = \sqrt{\frac{\mathbf{p}^2}{2}\left(\frac{\mathbf{p}^2}{2} + 2\varrho g\right)},\ \forall \mathbf{p} \in \Lambda_l^*\setminus\{\mathbf{0}\}$$

and

$$\xi_l(\mathbf{p}) = \frac{\mathbf{p}^2}{2} + g\varrho_{\mathbf{0}},\ \forall \mathbf{p} \in \Lambda_l^*\setminus\{\mathbf{0}\}.$$

This is, we have a system of non-interacting bosons with energies given by the above mentioned Bogoliubov spectrum. Moreover, $\hat{b}_{\mathbf{p}}\Psi_{\mathbf{0}} = 0$, being $\Psi_{\mathbf{0}}$ the ground state of the original system, i.e., excitations are absent from the ground state.

On the other hand,

$$\epsilon_l(\mathbf{p}) \approx \begin{cases} c||\mathbf{p}|| & \text{if } ||\mathbf{p}|| \ll \sqrt{2g\varrho}\ , \\ \frac{\mathbf{p}^2}{2} & \text{if } ||\mathbf{p}|| \gg \sqrt{2g\varrho}\ , \end{cases}$$

where $c = \sqrt{g\varrho}$ is the velocity of sound in liquid Helium.

This analysis, obviously, has been done in the framework of the canonical ensemble. The grand canonical approach considers an energy operator similar to the Hamitonian $\hat{H}_l'$ except that the total density must be replaced by the density of particles in the ground state. In this case the grand canonical Hamiltonian written in terms of quasiparticles, conserves their number, although this law is approximate since operators containing high order terms violate this conservation law [125,126].

In fact, the operators of creation and annihilation of quasiparticles $\hat{b}^{\dagger}_{\mathbf{p}}, \hat{b}_{\mathbf{p}}$ give account of the exchange of particles with the ground state.

Lev Landau proved, theoretically, that this kind of spectrum is associated to superfluidity, i.e., to the absence of disipation. Moreover it indicates the presence of two interpenetrating fluids, a superfluid one and a normal component constituted by quasiparticles (phonons). Although the Bogoliubov spectrum of excitations dominates absolutely the thermodynamics at low temperatures it does not describe the system behaviour at higher temperatures, where other quasiparticles, the rotons, drive thermodynamics.

> Bogoliubov's work has revealed the possibility of a non-perturbative approach to degenerate systems of bosons. It has led to the recognition of the important role of the condensate in establishing the phonon spectrum [139]

17. c-substitution and the approximating Hamiltonians method

An effective Hamiltonian could inherit or not inherit the conservation laws of the original energy operator, as is often the case with the mean field approximations in which the role of fluctuations is neglected, however, it is fundamental that their characteristics ensure the thermodynamic equivalence of the systems they represent in an adequate sense and that also allow to obtain information about correlation functions, order parameters, etc. This is the essence of the so-called approximating Hamiltonian method which usually complements quite well with the concept of quasimeans. Thus, for example, the usual strategy used for connecting BEC with spontaneous symmetry breaking consists in constructing an effective Hamiltonian depending on a complex parameter, in such a way that the pressures obtained by using the original operator and the effective one, respectively, coincide in the thermodynamic limit, in some sense, being the latter mathematically tractable. In other words, the original system is being compared with a solvable model. This fact makes possible the derivation of analytical expressions for limit thermodynamic functions.

Despite it was in 1947, in the context of the study of superfluidity in Bose gases at low density, and under a heuristic reasoning, that N. N. Bogoliubov [38] proposes that the operators $\hat{a}_{\mathbf{0}}^{\dagger}, \hat{a}_{\mathbf{0}}$, associated to the zero mode in the original Hamiltonian $\hat{H}_l(\mu)$ can be replaced by c-numbers for V_l large enough, i.e., $\hat{a}_{\mathbf{0}}^{\dagger} \to \sqrt{V_l}\bar{z}, \;\; \hat{a}_{\mathbf{0}} \to \sqrt{V_l}z$, obtaining an approximating energy operator $\hat{H}_l^{\mathrm{app}}(z,\mu)$, it was only in 1968 that J. Ginibre rigorously proves that the limit pressure of the weakly Bose gas is asymptotically equivalent to the pressure constructed by using such an approximation [39]. Moreover, from his proof it follows that this is a feasible replacement independent on the presence of condensate. Thus, the original justification of the c-number substitution based on the existence of macroscopic occupation of the zero mode showed to be erroneous.

Remember that, from a mathematical point of view, the Fock space $\mathcal{F}_B$ is decomposed as the tensorial product $\mathcal{F}_{\mathbf{0}} \otimes \mathcal{F}'_B$, where $\mathcal{F}_{\mathbf{0}}$ is the symmetric tensor algebra constructed in the one-dimensional space of constants (mode $\mathbf{0}$) and $\mathcal{F}'_B$ is its complementary orthogonal tensor algebra of functions in $L^2(\Lambda_l)$.

For any complex number $z \in \mathbb{C}$, let us consider a coherent vector in $\mathcal{F}_{\mathbf{0}}$, $\phi_z = e^{-V_l|z|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(V_l^{\frac{1}{2}} z\right)^n \left(a_{\mathbf{0}}^{\dagger}\right)^n \phi_{\mathbf{0}}$, being $\phi_{\mathbf{0}}$ the vacuum vector in $\mathcal{F}_{\mathbf{0}}$. Thus, for any operator$\hat{A}$ (with Wick ordering) defined on $\mathcal{F}_B$, it is possible to construct the so-called approximation of Bogoliubov $\hat{A}(z)$ in the following way

$$\left\langle \Psi'_1, \hat{A}(z)\, \Psi'_2 \right\rangle_{\mathcal{F}'_B} := \left\langle \Psi'_1 \otimes \phi_z, \hat{A}\Psi'_2 \otimes \phi_z \right\rangle_{\mathcal{F}_B},$$

where $\Psi'_1, \Psi'_2 \in \mathcal{F}'_{\mathbf{B}}$. In this sense, the transition from the operator $\hat{A}$ to the operator $\hat{A}(z)$ consists in replacing the operators $\hat{a}_{\mathbf{0}}$ y $\hat{a}_{\mathbf{0}}^{\dagger}$ for the complex numbers $\sqrt{V_l}z$ and $\sqrt{V_l}\bar{z}$, respectively in the operator $:\hat{A}:$ (normal form of $\hat{A}$). The standard proof of equivalence of pressures is based on a well known trace and convexity inequalities (see ref. [85]). Defining,

$$\chi_l(z,\mu) = \frac{\hat{H}_l^{\text{app}}(z,\mu) - \hat{H}_l(\mu)}{V_l}, \ \Delta_l(z,\mu) = \langle \chi_l(z,\mu) \rangle_{\hat{H}_l^{\text{app}}(\mu)},$$

it follows that $0 \leq p_l(\beta,\mu) - p_l^{\text{app}}(\beta,z,\mu) \leq \Delta_l(z,\mu)$.

Finally, if $\lim\limits_{V_l \to \infty} \inf\limits_z \Delta_l(z,\mu) = 0$, noting that:

$$0 \leq p_l(\beta,\mu) - \sup_z p_l^{\text{app}}(\beta,z,\mu) \leq \inf_z \Delta_l(z,\mu),$$

we get

$$p(\beta,\mu) = \lim_{V_l \to \infty} \sup_z p_l^{\text{app}}(\beta,z,\mu).$$

Thus,

> The validity of substituting a c-number z for the $\mathbf{p} = \mathbf{0}$ mode operator $\hat{a}_{\mathbf{0}}$ is established rigorously in full generality, thereby verifying one aspect of Bogoliubov 1947 theory. This substitution not only yields the correct value of thermodynamic quantities like the pressure or ground state energy, but also the value of $|z|^2$ that maximizes the partition function equals the true amount of condensation in the presence of a gauge-symmetry breaking term - a point that had previously been elusive [40]

18. Upper and lower bounds methods

18.1. **Bogoliubov inequalities.** Let $\hat{H}_{a,l}$ and $\hat{H}_{b,l}$ be self-adjoint operators defined on $\mathcal{D} \subset \mathcal{F}_B$. $p_{a,l}(\beta,\mu)$, $p_{b,l}(\beta,\mu)$ represent the grand canonical pressures and the free canonical energies corresponding to the operators $\hat{H}_{a,l}$, $\hat{H}_{b,l}$. In this case the following well known Bogoliubov inequalities,

$$\left\langle \frac{\hat{H}_{a,l}(\mu) - \hat{H}_{b,l}(\mu)}{V_l} \right\rangle_{\hat{H}_{a,l}(\mu)} \leq p_{b,l}(\beta,\mu) - p_{a,l}(\beta,\mu)$$
$$\leq \left\langle \frac{\hat{H}_{a,l}(\mu) - \hat{H}_{b,l}(\mu)}{V_l} \right\rangle_{\hat{H}_{b,l}(\mu)},$$

hold, where $\langle - \rangle_{\hat{H}_{a,l}(\mu)}$, $\langle - \rangle_{\hat{H}_{a,l}(\mu)}$ are the Gibbs states in the grand canonical ensemble associated to the Hamiltonians $\hat{H}_{a,l}$, $\hat{H}_{b,l}$, respectively.

18.2. **Infrared bounds method.** The derivation of upper and lower bounds for correlation functions to analyze classical and quantum phase transitions has been an essential part of statistical mechanics, mainly because of the difficulty of accurately solve the problem of determining exact expressions for limit grand canonical pressures. During the seventies of the past century, J. Frölich, B. Simon and T. Spencer developed the so-called Method of Infrared Bounds devoted to prove rigorously phase transition for some classical systems with continuous symmetries, by obtaining upper bounds for suitable correlation functions [59,86]. This strategy is based on notions such as Gaussian Domination and Reflection Positivity (RP). RP was introduced by K. Osterwalder and R. Schrader in ref. [87]. This approach was extended to quantum lattice spin systems by E. Lieb and B. Simon [62]. In the quantum case, the mentioned upper bounds are derived by using the so-called Falk-Bruch inequality [106], which is just one in a long list (see for example Brooks Harris inequality [109], Ginibre inequality [108], etc. See also ref. [111] for further reading) This fact makes necessary to derive, previously, upper bounds for the so-called Bogoliubov inner product or Duhamel two point function (DTPF) and for the thermal average of a double commutator. Bounds for the DTPF, under suitable conditions, can be obtained from a trace inequality-Gaussian Domination (GD)-, involving thermodynamic sums. In turn, GD follows in the case of quantum lattice systems, when possible, from symmetry arguments related to the algebra of observables.

> In nature, we observe abrupt changes in certain basic physical quantities, such as the magnetization of a magnet, but the statistical mechanics of systems with finitely many degrees of freedom is typically real analytic in all external variables. The resolution of this apparent paradox is that the abrupt changes are only approximately abrupt: True discontinuities only occur in the limit of an infinite system. For this reason, one must expect the problem of rigorously proving the existence of phase transitions to be a difficult one even for systems for which there is considerable numerical evidence or even a heuristic explanation for such a transition. [90]

The Bogoliubov inner product $(X,Y)_\Gamma$ is a bilinear form given by

$$(X,Y)_\Gamma = \left(\mathrm{Tr}\, e^{-\beta\Gamma}\right)^{-1} \int_0^1 \left(\mathrm{Tr}\, X e^{-\beta(1-x)\Gamma} Y e^{-\beta x\Gamma}\right) dx$$

where X, Y are arbitrary operators and Γ represents a selfajoint operator defined on a Hilbert space. This form commonly is associated to upper bounds for thermal averages of the type $\langle XX^* + X^*X\rangle_\Gamma$.

By introducing the following functions

$$g(A) = \frac{1}{2}\langle AA^* + A^*A\rangle_\Gamma,\ b(A) = (A^*, A)_\Gamma,$$
$$c(A) = \langle [A^*, [\beta H, A]]\rangle_\Gamma$$

for arbitrary operators A and assuming that $b, g, c > 0$ and $b \leq b_0$, $c \leq c_0$ it was proved that (Falk-Bruch inequality [106]):

$$g_0 = \tfrac{1}{2}\sqrt{c_0 b_0}\coth x_0,\ x_0 = \sqrt{\tfrac{c_0}{4b_0}}$$

18.3. **Lower bounds.** The following Bogoliubov inequalty,

$$\frac{1}{2}\langle\{A, A^*\}\rangle_\Gamma \langle [C^*, [\beta H, C]]\rangle_\Gamma \geq |\langle [C, A]\rangle_\Gamma|^2$$

provides a lower bound for the fluctuations of an arbitrary operator A in terms of C, where $\{A^*, A\}$ is the anticommutator defined as: $\{A^*, A\} = A^*A + AA^*$. A fundamental role is played by this inequality in proving the absence of long range order for Bose systems with a broken symmetry ($U(1)$ symmetry) in one and two dimensions in the framework of quasi-averages approach. In this case the so-called Hohenberg theorem [80] follows, by considering $A = \hat{a}_{\mathbf{p}}$ and $C = \hat{\rho}_{\mathbf{q}} = \sum_{\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}-\mathbf{q}}$ in the Bogoliubov inequality (see next section).

However, the Bogoliubov inequality is useless in the case of zero temperature. In such a case, L. Pitaevskii and S. Stringary derived the following inequality [105]:

$$\langle \{A, A^*\} \rangle_\Gamma \geq \tfrac{|[A^*, C] \rangle_\Gamma|^2}{\langle [C^*, C]] \rangle_\Gamma}.$$

Note that this inequality does not contain explicitly either the temperature or the Hamiltonian $\hat{H}$.

Finally, G. Roepstorff proved a more strong version of the Bogoliubov inequality [108]:

$$\langle A^* A \rangle_\Gamma \geq \langle [A, A^*] \rangle_\Gamma \exp \frac{\langle [C^*, [\beta H, C]] \rangle_\Gamma \langle [A, A^*] \rangle_\Gamma}{|\langle [A, C^*] \rangle_\Gamma|^2}.$$

The derivation of these kind of inequalities must be counted as a great step forward.

18.4. **The Hohenberg theorem.** In 1966, N. D. Mermin and H. Wagner demonstrated the absence of ferromagnetism or antiferromagnetism (long-range order) in one- or two-dimensional isotropic Heisenberg models [140]. The proof is based, precisely, on the mentioned inequality. In 1967, Hohenberg proved [80], by using the same inequality, the absence of lange range order in Bose and Fermi systems for one and two dimensions and finite nonzero temperatures. In 1969, N. Mermin published an article about some applications of Bogoliubov inequality in equilibrium statistical mechanics [142]. Beginning at this point, the use of such an inequality to demonstrate similar results in the case of different types of low dimensional particle systems became a usual scientific practice (see for example [107] and [143-146] for further reading).

> In the summer of 1965, Geoffrey Chester and I were in the Canadian Rockies, talking loudly to scare off grizzlies. Chester shouted that he had recently heard from Pierre Hohenberg that a curious inequality of Bogoliubov could be used to make an apparently rigorous proof that Bose-Einstein condensation or superconductivity could not happen in one or two dimensions. The Bogoliubov result appeared as an original article in a journal otherwise devoted to German translations of Russian papers. It was thus available in virtually no libraries outside of Germany, and I don't remember thinking further about the matter until fall. N.D. Mermin [141]

Let me sketch briefly the demonstration of absence of Bose-Einstein condensation in one and two dimensions given by Hohenberg.

Let $\Lambda_l^* = \{\mathbf{p} = (p_1, p_2, \cdots, p_d) \in \mathbb{R}^d\}$ be the set of wave vectors compatible with periodic boundary conditions, such that, $||\mathbf{p}|| \leq p_c$, for a system of Bose particles confined in a region of $\mathbb{R}^d$, with volume V. In the framework of the quasiaverage notion, the energy operator of an interacting Bose system is given by,

$$\hat{H}_{l,\nu} = \sum_{\mathbf{p}\in\Lambda_l{}^*} \frac{\mathbf{p}^2}{2} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{1}{2V_l} \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}\in\Lambda_l^*} \hat{U}(\mathbf{q}) \hat{a}_{\mathbf{p}+\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}'-\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{p}'}$$

$$-\sqrt{V}\nu(e^{-i\varphi}\hat{a}_{\mathbf{0}}^\dagger + e^{i\varphi}\hat{a}_{\mathbf{0}}),$$

with $\hat{U}(\mathbf{q}) = \hat{U}(-\mathbf{q})$, including a breaking symmetry term.

Let $A = \hat{a}_{\mathbf{q}}$ and $C = \hat{\rho}_{\mathbf{q}} = \sum_{\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}-\mathbf{q}}$. Besides, note that $\hat{\rho}^{*}_{\mathbf{q}} = \hat{\rho}_{-\mathbf{q}} = \sum_{\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}+\mathbf{q}}$.

By using the following identities,

$$\{A^{*}, A\} = 2\hat{n}_{\mathbf{q}} + 1,\ [C^{*}, [\beta \hat{H}_{l,\mu}(\mu), C]] = \beta \hat{N} \mathbf{q}^{2},\ [A, C] = \hat{a}^{\dagger}_{\mathbf{0}}$$

in the Bogoliubov inequality we get,

$$\langle \hat{n}_{\mathbf{q}} \rangle_{\hat{H}_{l,\nu}} \geq \frac{|\, \langle \hat{a}_{\mathbf{0}} \rangle_{\hat{H}_{l,\nu}} \,|^{2}}{\beta \left\langle \hat{N} \right\rangle_{\hat{H}_{l,\nu}} \mathbf{q}^{2}} - \frac{1}{2} = \frac{\eta_l^{2}}{\beta \rho_l \mathbf{q}^{2}} - \frac{1}{2}, \quad \text{for } \ \mathbf{q} \neq \mathbf{0},$$

where

$$\eta_l = \left| \left\langle \frac{\hat{a}_{\mathbf{0}}}{\sqrt{V}} \right\rangle_{\hat{H}_{l,\nu}} \right|.$$

This leads to,

$$\frac{1}{V} \sum_{\mathbf{q} \neq \mathbf{0}} \langle \hat{n}_{\mathbf{q}} \rangle_{\hat{H}_{l,\nu}} \geq \frac{1}{V} \sum_{\mathbf{q} \neq \mathbf{0}} \left(\frac{\eta_l^{2}}{\beta \rho_l \mathbf{q}^{2}} - \frac{1}{2} \right).$$

Being $\eta = \lim_{V \to \infty} \eta_l$, taking the limit $\nu \to 0$ after passing to the thermodynamic limit, the following inequality

$$\rho - \rho_{\mathbf{0}} \geq \lim_{\epsilon \to 0^{+}} \frac{1}{(2\pi)^{d}} \int_{\epsilon < ||\mathbf{q}|| < p_c} \left(\frac{\eta^{2}}{\beta \rho \mathbf{q}^{2}} - \frac{1}{2} \right) d\mathbf{q}$$

holds, where $d\mathbf{q} = dq_1 dq_2 \cdots dq_d$, and $\mathbf{q}^2 = \mathbf{q}\cdot\mathbf{q}$, being "$\cdot$" the usual euclidean inner product.

In this case, it is easy to see that,

$$\lim_{\epsilon\to 0^+}\int_{\epsilon<||\mathbf{q}||<\infty}\frac{d\mathbf{q}}{\mathbf{q}^2} = \begin{cases} \infty & \text{if } d = 1,2 \\ <\infty & \text{if } d \geq 3 \end{cases}$$

Thus, the infrared divergence can not be eliminated in one or two dimensions. Therefore, since the density of particles is a constant, the last inequality in such cases holds only if $\eta = 0$, i.e., in absence of continuous symmetry breaking..

> The Hohenberg theorem (HT) provides the general statement that Bose-Einstein condensation (BEC) cannot occur in a two-dimensional system. In this analysis a condensation implies extensive occupation of a single state of the system, that is, a density of particles of order N/V, (where N is the number of particles in the system and V the volume of the system) in the thermodynamic limit. Recently however there has been renewed interest in the possibility of a "smeared," "fragmented," or "generalized" BEC, in which some finite band of states, rather than a single state, is occupied.[133]

19. Large deviation method

There exists an extensive literature devoted to the study of mean field type Bose models. We shall restrict ourselves to mention just the basic ones. In the mean field model, the interaction is represented by the term $\frac{a\hat{N}^2}{2V}$ (Huang- Davis model), where the constant $a > 0$ represents the interaction intensity. In refs. [71,72] it was shown that, under suitable conditions, it represents a repulsive interaction, which at low densities is given by the potential $U(\mathbf{x}, \mathbf{y}) = a\delta(\mathbf{x} - \mathbf{y})$ or $\hat{U}(\mathbf{p}) = a$, for all $\mathbf{p} \in \mathbb{R}^3$, being $\hat{U}(\mathbf{p})$ the Fourier transform of $U(\mathbf{x}-\mathbf{y})$. The natural generalization was proposed by Davies in ref.[73].

$$\hat{H}_l^{HD} = \sum_{\mathbf{p}\in\Lambda_l{}^*} \lambda_l(\mathbf{p})\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + V_l g\left(\frac{\hat{N}}{V}\right),$$

where $g(x)$ is a positive convex and increasing function of $x \in [0, \infty)$, $a > 0$, and $\mathcal{F}_{\mathrm{B}}$ is the Fock space. Being the corresponding Hamiltonians diagonal operators with respect to the number operators, E. B. Davies gave a rigorous mathematical treatment to them on the basis of a probabilistic approach at infinite volume, obtaining an exact expression for the limit mean density of particles as function of the chemical potential. Unlike the free Bose gas, it was also proved that the imperfect Bose gas has the same limit thermodynamic behavior for both the microcanonical as the grand canonical ensembles, being also stable to the small perturbations influence.

In refs. [74-76] the *Method of Large Deviations* (LDM), based on a Varadhan theorem [77], was applied to the study of the so-called Huang-Davies and Huang-Davies-Luttinger model. The latter was introduced by Huang et.al. as a result of the application of the first order perturbation theory to general models. This has an interaction described by the operator:

$$\hat{H}_l = \tfrac{a}{2V}\left(2\hat{N}^2 - \sum_{j\geq 1}\hat{n}_j^2\right)$$

The LDM constitutes a powerful mathematical tool for rigorously proving the existence of the limit pressure and for deriving explicit expressions for it as function of chemical potential. In this case, the partition function can be written as an integral respect to a probability measure belonging to the set of bounded positive measures on $\mathbb{R}^+$.

Let the sequence $\{\Lambda_l : l = 1, 2, ..\}$ of regions in $\mathbb{R}^\nu$ and V_l. their respective volumes,. Each region Λ_l is associated a countable set Ω_l of configurations of particles enclosed in Λ_l.

Ω_l is a suitable probability space of random variables ω. In this framework, the selfadjoint operators $\hat{H}_l^{HD}$, $\hat{H}_l^0$ are replaced by the functions H_l^{HD} , $H_l^0 : \Omega_l \to \mathbb{R}$, respectively, interpreted as the energies of the Huang-Davies model and the non interacting model for the configurations ω and $N_l : \Omega_l \to \mathbb{N}$ represents the number of particles in Λ_l. Thus, the grand canonical measure $\mathbb{P}_l^\alpha$ associated to the free boson gas, with chemical potential μ, is defined on the subsets Ω_l as:

$$\mathbb{P}_l^\alpha[A] = [\Xi_l^{(0)}(\beta, \alpha)]^{-1} \sum_{\omega \in A \subset \Omega_l} e^{\beta(\alpha N_l(\omega) - H_l^0(\omega))}$$

where, as always, β is the inverse temperature, $\alpha < 0$, and $\Xi_l(\beta, \mu)$ is the grand canonical partiton function

$$\Xi_l(\beta, \mu) = \sum_{\omega \in \Omega_l} e^{-\beta V_l (g_\alpha \circ X(\omega))} e^{\beta(\alpha N_l(\omega) - H_l^0(\omega))},$$

$g_\alpha(x) = g(x) - (\mu - \alpha)x$, and the grand canonical pressure $p_l(\beta, \mu)$ is given as

$$p_l^{HD}(\beta, \mu) = p_l^0(\beta, \alpha) + \tfrac{1}{\beta V_l} \ln \sum_{\omega \in \Omega_l} e^{-\beta(g_\alpha \circ X(\omega))} \mathbb{P}_l^\mu[\omega].$$

By introducing the function $X_l = \frac{N_l}{V_l}$, the above relation can be rewritten as:

$$p_l^{HD}(\beta, \mu) = p_l^0(\beta, \alpha) + \tfrac{1}{\beta V_l} \ln \int_{[0,\infty)} e^{-\beta V_l g_\alpha(x)} \mathbb{K}_l^\mu(x),$$

where $\mathbb{K}_l^\alpha$ is the distribution function defined by $\mathbb{K}_l^\alpha = \mathbb{P}_l^\alpha \circ X_l^{-1}$.

There exists a function $I^\alpha(\cdot) : [0, \infty) \to [0, \infty)$, satisfying $\mathbb{K}_l^\alpha[dx] \sim e^{\beta V_l I^\alpha(x)}$ and denominated *rate function*, such that:

$$p^{HD}(\beta, \mu) = \lim_{V_l \to \infty} \frac{1}{\beta V_l} \ln \int_{[0,\infty)} e^{-\beta V_l g_\alpha(x)} \mathbb{K}_l^\alpha[dx]$$
$$= \sup_{x \in [0,\infty)} \{-g_\alpha(x) - I^\alpha(x)\} = \sup_{x \in [0,\infty)} \{(\mu - \alpha)x - g(x) - I^\alpha(x)\}.$$

This is nothing more than a singular version of the old principle of Laplace.

Finally, for the sequence $\{\mathbb{K}_l, l = 0, 1, 2, ..\}$ of probability measures on $[0, \infty)$ the rate function $I^\alpha(\cdot)$ is given as:

$$I^{\alpha}(x) = p^0(\beta, \alpha) + f^0(\beta, x) - \alpha x,$$

where $f^0(\beta, x)$ is the limit free energy of the free boson gas, obtained as the Legendre transform of $p^0(\beta, \cdot)$:

$$f^0(\beta, x) = \sup_{\alpha<0}\{\alpha x - p^0(\beta, x)\}.$$

20. Non-conventional BEC

20.1. Boundary conditions and spectral gap. The behavior of Bose systems at the boundary of its confinement region of volume V (for example, a cubic box of side l) is a fundamental problem in the study of many particle systems. In the case of a cubic box, the particles can be repelled, attracted or reflected by the boundaries.

Remember that the energy of a gas of non-interacting Bose particles is described by the free Laplacian operator $\triangle_l$ under suitable boundary conditions and its eigenvalues E, representing the energy levels of the system, are obtained from the equation $-\triangle_\ell \phi/2 = E\phi$, where ϕ are the free Laplacian eigenfunctions.

Different kinds of boundary conditions have been extensively studied for the free Bose gas for example in refs.[160-162].

In this case, we are interested in attractive boundary conditions which are given by the condition

$$\frac{\partial \phi}{\partial n} + \chi\phi = 0,$$

where $\partial\phi/\partial n$ is the outward normal derivative on the boundary, being $\chi < 0$ the attraction strength parameter. In this scenario, the spectrum of the Laplacian presents a gap. In other words, the lowest energy (which is $-d\chi^2$ in the thermodynamic limit, where d is the dimension of the system) is a non-vanishing (negative) isolated point of the spectrum. This fact, unlike Neumann, Dirichlet and periodic boundary conditions leads to condensation in any dimensions. Indeed, when

$\chi < 0$ we are in presence of a not homogeneous condensate mostly concentrated near the boundaries, i.e. many of its particles bind to the container walls. Thus the attractive wall condensation is a surface effect rather than the usual bulk phenomenon obtained under other boundary conditions. In realistic scenarios such as a not homogeneously distributed in space trapped gas the excitation spectrum is separated from the ground state [15]. Moreover, models with attractive boundary conditions are extremely useful in the study of wetting phase transitions in a superconductor [163] and for proving the existence of super-surface films in liquid helium II [11]. On the other hand, in the homogeneous cases any spectral gap vanishes in the thermodynamic limit.

Many authors have considered the idea of introducing a spectral gap between the ground state and the excited states in the case of some Bose gases, including the ideal one, for studying condensation in different situations.

In this sense, in refs. [164-166] a homogeneous Bose gas with periodic boundary conditions and a two-body interaction enclosed in a region of volume V has been exhaustively studied. In this works the term $-\triangle\hat{a}_{\mathbf{0}}^{\dagger}\hat{a}_{\mathbf{0}}$, where $\triangle > 0$ and $\hat{a}_{\mathbf{0}}^{\dagger}, \hat{a}_{\mathbf{0}}$ are the creation and annihilation operators of the zeroth mode, was added to the corresponding energy operator, shifting in this way the zeroth energy level of the kinetic energy operator to a negative value and generating a gap in the spectrum. Thus, it was proved the emergence of BEC at low temperatures for a class of superstable interacting systems at low enough temperatures, large enough particle densities and spectral gap.

For example, for the superstable mean field type model whose energy operator is given by

$$\hat{H}_b^{\triangle} = \hat{H}^{0,\triangle} + \frac{b}{2V}\hat{N}^2,$$

where $b > 0$, V is the volume enclosing the particle system, $\hat{H}^{0,\triangle}$ and $\hat{N}$ are the free energy operator including a spectral gap and the total number operator, respectively, in ref. [165] it is derived the following expression for the limit pressure $p_b^{\triangle}(\beta,\mu)$ of the model:

$$p_b^{\triangle}(\beta,\mu) = \begin{cases} p_b^{\triangle=0}(\beta,\mu), & \mu \leq -\triangle + b\rho^{\mathrm{p}}(\beta,-\triangle), \\ \frac{(\mu+\triangle)^2}{2b} + p^{\mathrm{p}}(\beta,-\triangle), & \mu > -\triangle + b\rho^{\mathrm{p}}(\beta,-\triangle), \end{cases}$$

where $p^{\mathrm{p}}(\beta,\mu)$ and $\rho^{\mathrm{p}}(\beta,\mu)$ (β–inverse temperature, μ–chemical potential) are the pressure and the total density of the ideal Bose gas, respectively. In this case the amount of condensate $\rho_{0,b}^{\triangle}(\beta,\mu)$ is given by:

$$\rho_{0,b}^{\triangle}(\beta,\mu) = \begin{cases} 0, & \mu \leq -\triangle + b\rho^{\mathrm{p}}(\beta,-\triangle), \\ \frac{\mu+\triangle}{b} - \rho^{\mathrm{p}}(\beta,-\triangle), & \mu > -\triangle + bp^{\mathrm{p}}(\beta,-\triangle). \end{cases}$$

The limit $\triangle \downarrow 0$ leads to

$$\rho_{0,b}^{\triangle=0}(\beta,\mu) = \begin{cases} 0, & \mu \leq b\rho_{\mathrm{c}}^{\mathrm{p}}(\beta), \\ \frac{\mu}{b} - p_{\mathrm{c}}^{\mathrm{p}}(\beta), & \mu > bp_{\mathrm{c}}^{\mathrm{p}}(\beta), \end{cases}$$

being $p_{\mathrm{c}}^{\mathrm{p}}(\beta)$ the critical density of BEC for the ideal gas. Clearly, the independent on temperature terms in the expressions for pressure and amount of condensate are consequence of the presence of the gap in the energy spectrum of the free energy operator. The amount of condensate still depends on temperature. However, noting that $\hat{N}^2 = \hat{N}'^2 + 2\hat{N}'\hat{n}_{\mathbf{0}} + \hat{n}_{\mathbf{0}}^2$, being $\hat{n}_{\mathbf{0}}, \hat{N}'$, the operator number of particles associated to the low energy level and the total number operator with exclusion of $\hat{n}_{\mathbf{0}}$, respectively, neglecting collisions between particles in the condensate and the rest of the system particles and assuming that the low energy level is equal to zero. For example, in the case of periodic boundary conditions we get a superstable model with energy operator

$$\hat{H}_b^{\triangle} = \hat{H}^{0,\triangle} + \frac{b}{2V}\hat{N}'^2,$$

for which macroscopic occupation of the ground does not occur. However, under attractive boundary conditions or by introducing a spectrum gap in the excitation spectrum of the free energy operator the above situation changes dramatically since an independent on temperature macroscopic occupation of the ground state takes place. In this sense, we are in presence of a different kind of condensation – non-conventional BEC (see, for example, works [167-173] for further details).

20.2. **Non conventional BEC.** In early papers we have been using different approaches such as large deviations and approximating Hamiltonian strategies [170-173] for obtaining the limit pressures in the case of Bose-atom systems with internal two level (spin-1/2) an one level (spin-0) structures. This enabled us to prove interesting results related to non-conventional or independent on temperature BEC in the case of Bose particle systems whose ground states are associated to negative values of energies. These results were extended to atoms systems with three internal levels and negative minimal energies [156].

Exactly solvable models of strongly interacting bosons could be helpful in understanding the nature of Bose-Einstein condensation (BEC) and superfluidity in interacting Bose gases. In ref. [156] the thermodynamic behavior of some theoretically relevant system of Bose particles was analyzed. The considered model has rather simplified character. Researches related to similar models are connected with the attempt to consider the effect of excitation-excitation coupling in Bogoliubov theory.

This system of Bose gas undergoes an independent on temperature BEC. This kind of BEC was theoretically discovered by Bru and Zagrebnov for some specific Bose system with diagonal interactions [167,168].

$$\hat{H}_l = \sum_{\mathbf{p}} \lambda_l(\mathbf{p}) \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{\gamma}{V^{k-1}} \sum_{\mathbf{p}} (\hat{a}^\dagger_{\mathbf{p}})^k \hat{a}^k_{\mathbf{p}}.$$

We study, using the Bogoliubov approximation, the thermodynamic behavior of a superstable Bose system whose energy operator in the second-quantized form contains a nonlinear expression in the occupation numbers operators. We prove that for all values of the chemical potential satisfying $\mu > \lambda(0)$, where $\lambda(0) \leq 0$ is the lowest energy value, the system undergoes Bose-Einstein condensation.

For the class of Hamiltonians given in the last equation, in the framework of the so-called Bogoliubov c approximation, it has been given a simple proof of thermodynamic equivalence of the limit grand-canonical pressures corresponding to those systems and their respective approximating ones for any integer $k \geq 2$. Moreover, in contrast to the Bru-Zagrebnov models, we prove that independent on temperature BEC in the sense of macroscopic occupation of the ground state holds for any integer $k \geq 2$ and any $\mu > \lambda(0)$. A similar type of BEC is explained entirely by superstability of the model and by absence of an interaction between the ground state occupation number operators and the nonzero modes ones.

By replacing the operators $\hat{a}_{\mathbf{0}}^{\dagger}$, $\hat{a}_{\mathbf{0}}$ in any operator A expressed in the normal form in the Hamiltonian $H_l(\mu)$ with the complex numbers $\sqrt{V}\bar{z}$ and $\sqrt{V}z$, $z \in \mathbb{C}$ we get the following approximating Hamiltonian,

$$\hat{H}_l = \sum_{\mathbf{p}\neq\mathbf{0}}(\lambda_l(\mathbf{p}) - \mu)\hat{a}_{\mathbf{p}}^{\dagger}\hat{a}_{\mathbf{p}} + \frac{\gamma}{V^{k-1}}\sum_{\mathbf{p}\neq\mathbf{0}}(\hat{a}_{\mathbf{p}}^{\dagger})^k\hat{a}_{\mathbf{p}}^k$$
$$+(\lambda(\mathbf{p}) - \mu)V|z|^2 + \gamma V|z|^{2k}.$$

In this case, the standard strategy leads to the following results for the limit pressure and the amount of condensate:

$$p(\beta,\mu) = \lim_{V_l\to\infty}\sup_{z\in\mathbb{C}} p_l^{\mathrm{app}}(\beta,z,\mu), \quad \rho_{\mathbf{0}} = \left(\frac{\mu - \lambda(\mathbf{0})}{\gamma k}\right)^{\frac{1}{k-1}}.$$

21. Non linear perturbation of the Ideal Bose Gas. M. Corgini and R. Tabilo

In this section we consider a model of a Bose gas whose energy operator corresponds to the sum of the Hamiltonian of the free Bose gas with a nonlinear perturbation represented by the square root of the number operator associated to the zero mode [175].

$$(1) \qquad \hat{H}^{\mathrm{app}}_{l,\nu}(\mu) = \sum_{\mathbf{p}\in\Lambda_l{}^*} \lambda_l(\mathbf{p})\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} - 2\nu\sqrt{V}\sqrt{\hat{n}_{\mathbf{0}}+1} - \mu\hat{N},$$

where $\nu > 0$.

The Hamiltonian given by eq. (2) represents a stable model defined in the domain $\mathcal{D} = \{(\beta,\mu) : \beta > 0,\ \mu < 0\}$.

On the other hand, let $\hat{H}^0_{l,\nu}(\mu)$ is defined as:

$$(2) \qquad \hat{H}^0_{l,\nu}(\mu) = \hat{H}^0_l(\mu) - \nu\sqrt{V}(e^{i\varphi}\hat{a}^\dagger_{\mathbf{0}} + e^{-i\varphi}\hat{a}_{\mathbf{0}}),\ \nu > 0.$$

Note that $[\hat{H}^{\mathrm{app}}_{l,\nu}(\mu), \hat{N}] = 0$, i.e. the energy operator given by eq.(2) preserves the $U(1)$ symmetry. However, $[\hat{H}^0_{l,\nu}(\mu), \hat{N}] \neq 0$, i.e., the continuous gauge symmetry associated with the $U(1)$ group is broken by the external field $-\nu\sqrt{V}(e^{i\varphi}\hat{a}^\dagger_{\mathbf{0}} + e^{-i\varphi}\hat{a}_{\mathbf{0}})$.

In the next section a strong connection between the critical behavior of both models, in the thermodynamic limit, will be established.

The main purpose of this work is to determine explicit expressions for the limit pressures of the model given by eq.(2) in the framework of the so called Laplace principle (see Appendix) and the Large Deviations Method based in two theorems proved by S. R. S. Varadhan [77]. Moreover, it shall be proven the existence of a phase characterized by the emergence of non conventional Bose-Einstein condensation, i.e., the existence of an independent on temperature condensate.

21.1. **Limit pressure and non conventional condensation.**

Theorem 21.1. *For $(\beta,\mu)\in\mathcal{D},\ \nu>0$, in the thermodynamic limit,*

$$p^{\mathrm{app}}(\beta,\mu,\nu) = -\frac{\nu^2}{\mu} + p^{\mathrm{id}'}(\beta,\mu), \tag{3}$$

where $p^{\mathrm{app}}(\beta,\mu,\nu), p^{\mathrm{id}'}(\beta,\mu)$ are the the limit pressures of the system whose Hamiltonian is given by the energy operator of eq. (2) and the energy operator given by eq. (1), but excluding the mode $\mathbf{0}$, respectively.

Proof. Let,

$$\hat{H}^{\mathrm{app}}_{l,\nu} = \sum_{\mathbf{p}\in\Lambda^*} \lambda(\mathbf{p})\hat{n}_{\mathbf{p}} - 2\sqrt{V}\nu\sqrt{\hat{n}_{\mathbf{0}}+1}.$$

Note that the function $h(x) = ax + b\sqrt{x+c},\ a,\ b \in \mathbb{R},\ x \in [0,\infty),\ c > 0$ is either an infinitely increasing or an infinitely decreasing mapping on $[0,\infty)$ except the case $a < 0,\ b > 0$.

Let $\{g_l\}$ be a sequence of functions defined on $[0,\infty)$ given as,

$$g_l(x) = (\mu - \lambda(\mathbf{0}))x + 2\nu\sqrt{x + \frac{1}{V}},\ x \in [0,\infty),$$

whose first and second derivatives are,

$$g_l'(x) = (\mu - \lambda(\mathbf{0})) + \nu\left(x + \frac{1}{V}\right)^{-1/2},$$

$$g_l''(x) = -\frac{\nu}{2}\left(x + \frac{1}{V}\right)^{-3/2} < 0,$$

respectively.

From these facts, it follows that $g_l(x)$ is a concave function attaining its global maximum at

$$x_l^* = \left(\frac{\nu}{\lambda((\mathbf{0}) - \mu}\right)^2 - \frac{1}{V},$$

for a large enough value of V such that $x_l^* \geq 0$ and

$$\lim_{V\to\infty} \sup_{x\in[0,\infty)} g_l(x) = \lim_{V\to\infty} \sup_{x\in[0,\infty)} g_l(x_l^*) = -\frac{\nu^2}{\mu},$$

being $\mu < 0$, $\lambda(\mathbf{0}) = 0$.

Use will be made of the so-called large deviations method, based on the Laplace principle, for obtaining a closed analytical expression for $p^{\mathrm{app}}(\beta,\mu,\nu)$. Since $\hat{H}^{\mathrm{app}}_{l,\nu}(\mu)$ is a diagonal operator with respect to the number operators, the finite pressure can be written as,

$$p_l^{\text{app}}(\beta,\mu,\nu) = \frac{1}{\beta V}\ln \operatorname{Tr}_{\mathcal{F}_B} \exp\{-\beta \hat{H}_{l,\nu}^{\text{app}}(\mu)\}$$

$$p_l^{\text{app}}(\beta,\mu,\nu)_l = \frac{1}{\beta V}\ln \sum_{n_{\mathbf{0}}=0}^{\infty} \exp\beta\{(\mu-\lambda(\mathbf{0}))n_{\mathbf{0}} + 2\sqrt{V}\nu\sqrt{n_{\mathbf{0}}+1}\}$$

$$+\frac{1}{\beta V}\ln \sum_{\mathbf{p}\in\lambda^*\setminus\{\mathbf{0}\},n_{\mathbf{p}}}^{\infty} \exp\beta\{(\mu-\lambda(\mathbf{p}))n_{\mathbf{p}}\}.$$

Noting, that

$$p_l^{\text{app},\mathbf{0}}(\beta,\mu,\nu)_l = \frac{1}{\beta V}\ln \sum_{n_{\mathbf{0}}=0}^{\infty} \exp\beta\{(\mu-\lambda(\mathbf{0}))n_{\mathbf{0}} + 2\sqrt{V}\nu\sqrt{n_{\mathbf{0}}+1}\}$$

$$= \frac{1}{\beta V}\ln \sum_{n_{\mathbf{0}}=0}^{\infty} \exp\beta V\left\{(\mu-\lambda(\mathbf{0}))\frac{n_{\mathbf{0}}}{V} + 2\nu\sqrt{\frac{n_{\mathbf{0}}}{V}+\frac{1}{V}}\right\}$$

$$= \frac{1}{\beta V}\ln \sum_{n_{\mathbf{0}}=0}^{\infty} \exp\left\{\beta V g_l\left(\frac{n_{\mathbf{0}}}{V}\right)\right\}.$$

It is not hard to see that $\{p_l^{\text{app},\mathbf{0}}(\beta,\mu,\nu)\}$ is a sequence of Darboux sums, then, in the thermodynamic limit the Laplace principle leads to the following expression,

$$p^{\text{app}}(\beta,\mu,\nu) = -\frac{\nu^2}{\mu} + p^{\text{id}'}(\beta,\mu).$$

$\square$

Theorem 21.2. *For $(\beta,\mu) \in \mathcal{D}$, $\nu > 0$, in the thermodynamic limit, the Bose Gas with Hamiltonian given by eq.(2) undergoes non conventional condensation if and only if, the ideal gas whose energy operator is given by eq. (1) also displays independent on temperature condensation. Moreover,*

$$p^{\mathrm{id}}(\beta,\mu,\nu) = p^{\mathrm{app}}(\beta,\mu,\nu), \tag{4}$$

and the amount of condensate satisfies:

$$\rho_{\mathbf{0}}^{\mathrm{app}}(\mu,\nu) = \rho_{\mathbf{0}}^{\mathrm{id}}(\mu,\nu) = \frac{\nu^2}{\mu^2} \tag{5}$$

Proof. Note that:

$$p_l^{\mathrm{id}}(\beta,\mu,\nu) = \frac{1}{\beta V} \ln\left(1 - e^{\beta\mu}\right) - \frac{\nu^2}{\mu} + p_l^{\mathrm{id}'}(\beta,\mu). \tag{6}$$

From this it follows that:

$$p_l^{\mathrm{id}}(\beta,\mu,\nu) - p^{\mathrm{app}}(\beta,\mu,\nu) = \frac{1}{\beta V} \ln\left(1 - e^{\beta\mu}\right) + p_l^{\mathrm{id}'}(\beta,\mu) - p^{\mathrm{id}'}(\beta,\mu).$$

Thus, in the thermodynamic limit, for fixed values of β and strictly negative values of μ,

$$p^{\mathrm{id}}(\beta,\mu,\nu) = p^{\mathrm{app}}(\beta,\mu,\nu).$$

On the other hand, using the Griffiths Lemma [27],

$$\partial_\mu p_l^{\mathrm{id}}(\beta,\mu,\nu) - \partial_\mu p^{\mathrm{app}}(\beta,\mu,\nu) = \rho_l^{\mathrm{id}}(\beta,\mu,\nu) - \rho^{\mathrm{app}}(\beta,\mu,\nu)$$
$$= \frac{1}{V}\left(\frac{1}{e^{-\beta\mu}-1}\right).$$

in this case,

$$\rho^{\mathrm{app}}(\beta,\mu,\nu) = \frac{\nu^2}{\mu^2} + \rho_c(\beta,\mu), \tag{7}$$

$$\rho_l^{\mathrm{id}}(\beta,\mu,\nu) = \frac{\nu^2}{\mu^2} + \frac{1}{V}\left(\frac{1}{e^{-\beta\mu}-1}\right) + \rho_{c,l}(\beta,\mu). \tag{8}$$

From eqs. (8) and (9) we get:

$$\rho_{\mathbf{0}}^{\mathrm{app}}(\beta,\nu,\nu) = \rho^{\mathrm{app}}(\beta,\mu,\nu) - \rho_c(\beta,\mu), \tag{9}$$

$$\rho_{\mathbf{0},l}^{\mathrm{id}}(\beta,\mu,\nu) = \rho_l^{\mathrm{id}}(\beta,\mu,\nu) - \rho_{c,l}(\beta,\mu), \tag{10}$$

being, as before:

$$\rho_{c,l}(\beta,\mu)=\frac{1}{V}\sum_{\mathbf{p}\in\Lambda_l{}^*\backslash\{0\}}\left(\exp\beta\left(\frac{\mathbf{p}^2}{2}-\mu\right)-1\right)^{-1},$$

$$\rho_c(\beta,\mu)=\frac{1}{(2\pi)^d}\int\left(\exp\beta\left(\frac{\mathbf{p}^2}{2}-\mu\right)-1\right)^{-1}d^3\mathbf{p}.$$

Since,

$$\lim_{V\to\infty}\frac{1}{\beta V}\left(\frac{1}{e^{-\beta\mu}-1}\right)=0,$$

for the fixed parameters $(\beta,\mu)\in\mathcal{D},\ \ \nu>0$, and from the expressions in eqs. (10) and (11), we have that both systems, simultaneously, undergo non conventional condensation. Moreover, the amount of condensate is given as:

$$\rho_{\mathbf{0}}^{\text{app}}(\mu,\nu)=\rho_{\mathbf{0}}^{\text{id}}(\mu,\nu)=\frac{\nu^2}{\mu^2}.$$

□

The Bogoliubov approach considers a chemical potential μ_* such that $\mu_*=-\frac{\nu}{\sqrt{\rho_{\mathbf{0}}}}$, being $\rho_{\mathbf{0}}$ a real and strictly positive constant

On the other hand, unlike the system given by the Hamiltonian in eq.(1), the system whose energy operator is represented by (2) preserves the U(1) symmetry.

If $\rho^{\text{id}}_{0,l}(\beta, \mu_l, \nu) = \rho_0 = \text{constant} \neq 0, \ \rho_0 > 0, \mu_l < 0$, we have that:

$$\rho_0 \sim \frac{1}{V\left(-\mu_l + \mu_l^2/2\right)} + \frac{\nu^2}{\mu_l^2}.$$

Thus, for values of μ_l in a small neighborhood of zero,

$$\beta V \rho_0 \sim -\frac{1}{\mu_l} + \frac{\beta V \nu^2}{\mu_l^2}.$$

By solving the second order equation in μ_l, we obtain:

$$\mu_l \sim -\frac{1}{2\beta V \rho_0}\left(1 + \sqrt{1 + (2\beta V \nu)^2 \rho_0}\right).$$

Finally, taking the thermodynamic limit:

$$\lim_{V\to\infty} \mu_l = \mu^* = -\frac{\nu}{\sqrt{\rho}_0}.$$

For the free Bose gas, this result means that in the domain $\mathcal{D}$, in spite of that the chemical potential μ_l depends on the inverse temperature β at finite volume, in the thermodynamic limit μ depends only on the fixed parameters ρ_0, ν.

21.2. **Full diagonal models.** Let $\hat{H}_l^{\text{FD}}(\mu)$ be the energy operator defined as:

$$\hat{H}_l^{\text{FD}}(\mu) = \hat{H}_l^0 + \frac{a}{2V}\left(\hat{N}^2 - \hat{N}\right) + \frac{1}{2V}\sum_{\mathbf{p},\mathbf{p}'} v(\mathbf{p}-\mathbf{p}')\hat{n}_{\mathbf{p}}\hat{n}_{\mathbf{p}'}. \tag{11}$$

$\hat{H}_l^{\mathrm{FD}}(\mu)$ belongs to a class of energy operators so-called *full diagonal Bose Hamiltonians*. Clearly, $\hat{H}_l^{\mathrm{FD}}(\mu)$ satisfies the commutation rule $[\hat{H}_l^{\mathrm{FD}}(\mu), \hat{N}] = 0$. For example, $\hat{H}_l^0$ is a full diagonal mode. If $a > 0$, and $v(\mathbf{p} - \mathbf{p}') \geq 0$ these are superstable systems, i.e., their limit pressures exist for all real value of μ.

Let $\hat{H}_{l,\nu}^{\mathrm{FD}}(\mu)$, $\hat{H}_{l,\nu}^{\mathrm{FD,app}}(\mu)$ be the following operators:

$$\hat{H}_{l,\nu}^{\mathrm{FD}}(\mu) = \hat{H}_l^{\mathrm{FD}}(\mu) - \nu\sqrt{V}(\hat{a}_{\mathbf{0}}^{\dagger} + \hat{a}_{\mathbf{0}}),\ \nu > 0, \tag{12}$$

$$\hat{H}_{l,\nu}^{\mathrm{FD,app}}(\mu) = \hat{H}_l^{\mathrm{FD}}(\mu) - 2\nu\sqrt{V}\sqrt{\hat{n}_{\mathbf{0}} + 1},\ \nu > 0. \tag{13}$$

In this case, $[\hat{H}_{l,\nu}^{\mathrm{FD,app}}(\mu), \hat{N}] = 0$, and $[\hat{H}_{l,\nu}^{\mathrm{FD}}(\mu), \hat{N}] \neq 0$.

Theorem 21.3.

$$p^{\mathrm{FD}}(\beta, \mu, \nu) = p^{\mathrm{FD,app}}(\beta, \mu, \nu). \tag{14}$$

Proof. For this kind of models in ref.[83] it has been proved that:

(15) $$\lim_{V\to\infty}\left\langle \frac{\hat{a}_{\mathbf{0}}^{\dagger}}{\sqrt{V}} \right\rangle_{\hat{H}_{l,\nu}^{\mathrm{FD}}(\mu)} = \lim_{V\to\infty}\left\langle \frac{\hat{a}_{\mathbf{0}}}{\sqrt{V}} \right\rangle_{\hat{H}_{l,\nu}^{\mathrm{FD}}(\mu)}$$
$$= \mathrm{sgn}\ \nu \lim_{\mathrm{V}\to\infty}\sqrt{\mathrm{V}^{-1}\left\langle \hat{\mathrm{a}}_{\mathbf{0}}^{\dagger}\hat{\mathrm{a}}_{\mathbf{0}} \right\rangle_{\hat{\mathrm{H}}_{\mathrm{l},\nu}^{\mathrm{FD}}(\mu)}}.$$

In our case sgn $\nu = +$.

Let define δ_{p_l}, y δ_H as

$$\delta_{p_l} = p_l^{\mathrm{FD}}(\beta,\mu,\nu) - p_l^{\mathrm{FD,app}}(\beta,\mu,\nu),$$

$$\delta_H = \hat{H}_{l,\nu}^{\mathrm{FD,app}}(\mu) - \hat{H}_{l,\nu}^{\mathrm{FD}}(\mu) = \nu\sqrt{V}\left(2\sqrt{\hat{n}_{\mathbf{0}}+1} - (\hat{a}_{\mathbf{0}}^{\dagger} + \hat{a}_{\mathbf{0}})\right),$$

respectively.

Note that $\hat{H}^{\mathrm{FD,app}}(\mu)$ preserves the $U(1)$ symmetry. This fact and the left hand side Bogoliubov inequality (see the Apendix) lead to:

$$\delta p_l \geq 2\nu\left\langle \sqrt{\hat{\rho}_{\mathbf{0},l} + \frac{1}{V}} \right\rangle_{\hat{H}_{l,\nu}^{\mathrm{FD,app}}(\mu)} \geq 0.$$

Moreover, in the thermodynamic limit we get,

(16) $$\lim_{V\to\infty}\delta_{p_l} =\geq 2\nu\lim_{V\to\infty}\left\langle \sqrt{\hat{\rho}_{\mathbf{0},l}} \right\rangle_{\hat{H}_{l,\nu}^{\mathrm{FD,app}}(\mu)} \geq 0.$$

From the right hand Bogoliubov inequality and the Jensen inequality (see Apendix) we obtain:

$$\delta p_l \leq \frac{\nu}{\sqrt{V}} \left\langle 2\sqrt{\hat{n}_{\mathbf{0}}+1} - (\hat{a}_{\mathbf{0}}^{\dagger} + \hat{a}_{\mathbf{0}}) \right\rangle_{\hat{H}_{l,\nu}^{\mathrm{FD}}(\mu)}$$

$$(17) \quad \leq \nu \left(2\sqrt{\hat{\rho}_{\mathbf{0},l} + \frac{1}{V}} - \left\langle \frac{\hat{a}_{\mathbf{0}}^{\dagger}}{\sqrt{V}} \right\rangle_{\hat{H}_{l,\nu}^{\mathrm{FD}}(\mu)} - \left\langle \frac{\hat{a}_{\mathbf{0}}}{\sqrt{V}} \right\rangle_{\hat{H}_{l,\nu}^{\mathrm{FD}}(\mu)} \right).$$

Finally, taking the limit $V \to \infty$ and using the expressions in eq.(16) and the inequalities (17) and (18),

$$0 \leq \lim_{V\to\infty} \delta p_l \leq 0,$$

hence $p^{\mathrm{FD}}(\beta,\mu,\nu) = p^{\mathrm{FD,app}}(\beta,\mu,\nu)$.

□

A well-known example of a full diagonal Hamiltonian is associated to the mean field model, whose energy operator with an additional term broken the U(1) symmetry, is given by the expression:

$$\hat{H}_{l,\nu}^{\mathrm{MF}} = \hat{H}^0 + \frac{a}{2V}\left(\hat{N}^2 - \hat{N}\right) - \nu\sqrt{V}(\hat{a}_{\mathbf{0}}^{\dagger} + \hat{a}_{\mathbf{0}}),$$

where $a > 0$, V is the volume of the region enclosing the particle system and $\nu \in \mathbb{R}$.

In this case, the operator $\hat{H}^{\mathrm{MF,app}}_{l,\nu}$ has the following form:

$$\hat{H}^{\mathrm{MF,app}}_{l,\nu} = \hat{H}^0 + \frac{a}{2V}\left(\hat{N}^2 - \hat{N}\right) - \nu\sqrt{V}\sqrt{\hat{n}_{\mathbf{0}} + 1}.$$

As before, $\nu > 0$.

21.3. **Conclusions.**

a. For fixed parameters $\mu < 0$, $\nu > 0$, the pressures and the density of particles in the condensates of the systems whose operators are given by eqs. (1) and (2), in the thermodynamic limit, coincide. Thus,

$$p^{\mathrm{app}}(\beta,\mu,\nu) = p^{\mathrm{id}}(\beta,\mu,\nu) = -\frac{\nu^2}{\mu} + p^{\mathrm{id}'}(\beta,\mu),$$

$$\rho^{\mathrm{id}}_{\mathbf{0}}(\mu,\nu) = \rho^{\mathrm{app}}_{\mathbf{0}}(\mu,\nu) = \frac{\nu^2}{\mu^2},$$

i.e., both models are equivalent in a thermodynamic sense and they undergo, simultaneously, non conventional BEC in $\mathcal{D}$.

b. The full diagonal models, with coupled external sources given in eqs. (13) and (14), are thermodynamically equivalent ones.

c. Despite what has been said in a) and b), the external source $-2\nu\sqrt{V}\sqrt{\hat{n}_{\mathbf{0}}+1}$ does not remove the degeneracy of the regular averages.

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22. Trapped gases and the uniform Bose gas

22.1. **Trapped gases.** Atoms, confined in a magmetic trap, are cooled by using laser cooling techniques and after this, by applying evaporative cooling methods, allowing the most thermal atoms escape from the trap, while the remaining ones thermalizes into low temperatures. Besides the fact that completely isolated finite systems do not exist [53], Bose trapped Gases shows the following features:

1. Trapped gases are finite size systems.
2. The number of atoms that can be put into the traps can not be considered macroscopic: 10^2-10^7 atoms. Indeed, this number of atoms is still far from what we consider macroscopic world (10^{23}).
3. In the framework of the standard theory of phase transitions, BEC and the corresponding breakdown of $U(1)$ symmetry exist only in the so-called thermodynamic limit (bulk system) for, at least, stable systems.
4. In the presence of a significant condensate, interaction effects becomes determinant and dominate at $T \ll T_c$. In this scenario the Bose ideal model does not describe adequately the system behavior.
5. Trapped systems are spatially inhomogeneous -at least until 2013 when BEC in a quasi uniform three dimensional potential of an optical trap box was studied.
6. Trapped Bose gases can display low dimensional BEC (in a mesoscopic sense), phenomenon prohibited for infinite Bose particle systems (Hohenberg theorem).

7. The thermodynamic limit is never reached exactly.
9. The number of particles is conserved implying that the global U(1) symmetry is preserved (if some symmetry is broken it has to be a global one, i.e., its corresponding order parameter must be independent on space-time, since from the so-called Elitzur′s theorem [115] it follows that only non local order parameters can be nonzero in the thermodynamic limit, although, it has to be pointed out that a global symmetry can be always obtained from a local symmetry by fixing the group′s parameters of the latter in position and time).
10. In spite of (9) the merging of two condensates produces interference patterns in their densities sometimes attributable to spontaneous rupture of the continuous U(1) symmetry. However, it has been a controversial assumption.
11. To overcome the difficulties in defining pressure and volume for a gas confined in an inhomogeneous trap, it is necessary to define macroscopic parameters that behave like them.

In this context, let us consider the case of an anisotropic gas confined in a magnetic trap with the harmonic potential [15]:

$$V_{ext}(\mathbf{r}) = \frac{m}{2}\left(\omega_x x^2 + \omega_y y^2 + \omega_z z^2\right).$$

The accessible energies are given by $E_{n_x n_y n_z} = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z$, where n_x, n_y, n_z take positive integer values.

On the other hand $\Phi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = \otimes_{i=1}^{N} \phi_0(\mathbf{r}_i)$ is the ground state for the $N-$ particle system, where

$$\Phi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = \otimes_{i=1}^{N} \phi_0(\mathbf{r}_i),$$

and

$$\phi_0(\mathbf{r}) = \left(\frac{m\omega_{h_0}}{\pi\hbar}\right)^{3/2} \exp\left[-\frac{m}{2\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)\right]$$

being $\omega_{h_0} := (\omega_x\omega_y\omega_z)^{\frac{1}{3}}$ the so-called geometric mean (effective frequency). For the energy $E_{000} = \frac{\hbar}{2}(\omega_x + \omega_y + \omega_z)$, the density distribution has the form: $n(\mathbf{r}) = N|\phi_0(\mathbf{r})|^2$. Finally the size of the cloud and the number of particles in the grand canonical ensemble are:

$$a_{h_0} = \int |\phi_0(\mathbf{r})|^2 d\mathbf{r} = \left(\frac{\hbar}{m\omega_{h_0}}\right)^{1/2},$$

$$N = \sum_{n_x, n_y, n_z} \left(e^{\beta(E_{n_x n_y n_z} - \mu)} - 1\right)^{-1},$$

respectively.

Thus,

$$N - N_0 = \sum_{n_x, n_y, n_z \neq 0} \frac{1}{e^{\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)} - 1},$$

for $\mu = \frac{1}{2}(\omega_x + \omega_y + \omega_z)$, being N_0 the number of particles in the condensate.

In the semiclassical approximation $\theta \gg \hbar\omega_{h_0}$ the above sums can be replaced by integrals:

$$N - N_0 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{dn_x dn_y dn_z}{e^{\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)} - 1} = \zeta(3)\left(\frac{1}{\beta\hbar\omega_{h_0}}\right)^3$$

being $\zeta(n)$ is the Riemann zeta function.

When $N_0 \to 0$ we get the following inverse critical temperature β_c for BEC transition:

$$\beta_c^{-1} = \theta_c = \hbar\omega_{h_0}\left(\frac{N}{\zeta(3)}\right)^{1/3} = C\hbar\omega_{h_0}N^{1/3}.$$

A kind of thermodynamic limit can be obtained by passing to the limits $N \to \infty$, $\omega_{h_0} \to 0$ and by keeping the product $\omega_{h_0}N^{1/3}$ constant.

For $\beta \geq \beta_c$,

$$\frac{N_0}{N} = 1 - \left(\frac{\beta_c}{\beta}\right)^3.$$

There are many differences between this kind of systems and the model of the uniform Bose gas. Indeed. in the latter case, the inverse temperature of transition and the ratio between the number of particles in the condensate and the total number of them in the system are given by the following expressions:

$$\beta_c^{-1} = \frac{2\pi\hbar}{m}\left(\frac{N}{V\zeta(3/2)}\right)^{2/3},$$

$$\frac{N_0}{N} = 1 - \left(\frac{\beta_c}{\beta}\right)^{3/2},$$

respectively.

22.2. **Uniform potential.** Despite that ultracold Bose gases have been traditionally used for testing fundamental many-body physics, there exist significant differences with “standard” many-body systems. Indeed, while these are uniform, the ultracold gases in harmonic traps do not have translational symmetries.

> In the magnetic traps, not only is the number of particles quite small, compared to the usual case, but the “boundary,” formed by a quadratic potential well, extends literally throughout the whole system. In order to take the thermodynamic limit in such a system it is necessary to weaken the potential so that, as the number of particles increases, the average density remains constant. This is well-defined mathematically, but is of course physically unrealizable. On the other hand, taking the box size to infinity in the homogeneous case is also unrealized experimentally. [112]

23. U(1) Symmetry breaking?

When passing the critical temperature for BEC a relative phase distribution between condensate and non condensate phases has been also experimentally detected. Obviously, since the number of particles is finite and preserved, these phenomena can not be produced by a spontaneous U(1) symmetry breaking. On the other hand, the usual external symmetry breaking field in the case of atoms should be constituted by atoms of the same sort and since the number of atoms is finite and well-known, the phase should be unknown as predicted by the so-called phase-number uncertainty relation [113]. Moreover, it does not seem possible to determine whether the interference pattern is an effect of the experimental observation or a phenomena previous to any observational analysis. Finally, in trapped BEC interference patterns arise also from the overlapping of independently Bose Einstein condensates.

24. Some comments

Ketterle and van Drutten proved that the results obtained in finite size sytems, in cases of certain dilute atomic gases, for some values of the critical parameters, such as chemical potential, temperature and condensate density, differs with those obtained in the thermodynamic limit. Moreover they shown that the occupation of states of low energies, for these parameters, in that limit, disappear [1,2].

The number of atoms that can be put into the traps is not truly macroscopic. So far experiments have been carried out with a maximum of about 10^7 atoms. As a consequence, the thermodynamic limit is never reached exactly.

A first effect of the above mentioned fact is the lack of discontinuities in the thermodynamic functions. Hence Bose-Einstein condensation in these trapped gases is not, strictly speaking, a phase transition. In practice, however, the macroscopic occupation of the lowest state occurs rather abruptly as temperature is lowered and can be observed. The transition is actually rounded with respect to the predictions of the $N \to \infty$ limit, but this effect, though interesting, is small enough to make the words transition and critical temperature meaningful even for finite-sized systems. It is also worth noticing that, instead of being a limitation, the fact that N is finite makes the system potentially richer, because new interesting regimes can be explored even in cases where there is no real phase transition in the thermodynamic limit.

In this sense, the traps can be made very anisotropic, reaching the limit of quasi-2D and quasi-1D systems, so that interesting effects of reduced dimensionality can be also studied, as two dimensional BEC for finite size systems. Indeed, BEC state is impossible for an interacting gas in a 2D trap in the thermodynamic limit, although finite size BEC holds [3,4].

In this scenario, the condensation of type III, before mentioned, where a macroscopically significant contribution of many states in a small band of energies near to zero produces, in the thermodynamic limit, an effect similiar to the macroscopic ocupation of single states, seems to be a fertil mathematical ground for further studies.

25. New frontiers

The so-called quantum vortices play a very important role not only in superfliudity but in superconductivity too. A quantum vortex is composed by a core of non superfluid matter surrounded by superfluid matter rotating respect to the first one. This is a well-defined stable topological defect.

Unlike of the liquid helium, having a strong interatomic interaction, the studied alkaline trapped gases show a weak interatomic interaction and therefore their connection with superfluity can be analyzed more easily. This fact prompted physicists to try to experimentally produce "vortices" in the case of Bose-Einstein gaseous condensates under rotation [91,92]. Indeed, the formation of highly ordered vortices lattices was confirmed in 2001 experimentally in a rotating Bose gas [93]. In this case, when rotating or agitating the condensate, groups of atoms emerge in closed circles, such as vortices or turbulent clouds, a situation also observed in the case of helium in 2009 [94].

In an experiment conducted in 1998 by researchers at Harvard University led by Danish researcher Lene Vestergaard Hau the speed of light passing through a condensate of sodium atoms was reduced to 17 meters per second. This result could have a huge impact on quantum information theory, especially if our interest is to store it.

On the other hand, the permanent interaction of photons with the medium should imply the non-conservation of their total number. For this reason, the possibility that they could condense was until recently disregarded. Indeed, when decreasing temperature the photons tend to disappear by the absorption in the walls of the confinement region. However, in 2010, in an experiment carried out at the University of Bonn, the induction of a special thermalization process allowed to preserve the number of photons in confinement with the consequent formation of a photonic condensate in two dimensions [96]:

26. Appendix

26.1. **Jensen's Inequality.** Let $\hat{H}_l$, be a self-adjoint operator, diagonal with respect to the number operators.Since the spectrum of $\hat{H}_l$, coincides with the set of non negative integers, this model can be classically understood by using non negative random variables defined on a suitable probability space Ω_l.

Let Ω_l be the countable set of sequences $\omega = \{\omega(p) \in \mathbb{N} : p \in \Lambda_l^*\} \subset \mathbb{N} \cup \{0\}$ satisfying

$$\sum_{p \in \Lambda_l^*} \omega(p) < \infty \ .$$

The basic random variables are the occupation numbers $\{n_p : j = 1, 2, ...\}$. They are defined as the functions $n_p : \Omega_l \to \mathbb{N}$ given as $n_p(\omega) = \omega(p)$ for any $\omega \in \Omega_l$. The total number of particles in the configuration ω is denoted as $N(\omega)$. Then the total number, excluded the zero mode is denoted as $N'(\omega)$.

In this framework, the Gibbs state can be written by replacing $\hat{H}_l$, by a function $H_l : \Omega_l \to \mathbb{R}$, representing the projection of the energy operator on the occupation-number basis of the Bose Fock space.

Let $\mathbb{P}$ be a probability defined for any $\omega \in \Omega_l$ as

(18)
$$\mathbb{P}[\omega] = \left[\sum_{\omega \in \Omega_l} \exp\left(-\beta[H_l(\mu)](\omega)\right) \right]^{-1} \exp\left(-\beta[H_l(\mu)](\omega)\right).$$

For arbitrary $S \subset \Omega$ this implies that

(19)

$$\mathbb{P}[S \subset \Omega_l] = \left[\sum_{\omega\in\Omega_l} \exp\left(-\beta[H_l(\mu)](\omega)\right)\right]^{-1} \sum_{\omega\in S} \exp\left(-\beta[H_l(\mu)](\omega)\right).$$

In this case, $\left\langle \hat{X} \right\rangle_{\hat{H}_l(\mu)} \equiv \mathbb{E}[X]$, being $X : \Omega_l \to \mathbb{R}$ the function corresponding to the proyection of the operator $\hat{X}$ on the occupation-number basis.

Thus, the expectation of X respect to $\mathbb{P}$ is defined as:

$$\mathbb{E}[X] = \sum_{\omega\in\Omega_l} X(\omega)\mathbb{P}[\omega]. \tag{20}$$

If $X : \mathbb{R} \to \mathbb{R}$ is a concave function, the following Jensen's inequality,

$$\mathbb{E}[f(X)] \leq f(\mathbb{E}[X]), \tag{21}$$

holds.

26.2. **Griffiths Lemma.**

Lemma 26.1. *(Griffiths [27] Let* $\{g_n : I \to \mathbb{R},\ I \equiv (a,b) \subset \mathbb{R}\}_{n\in\mathbb{N}}$ *be a sequence of convex functions on* I *with a pointwise limit* $g(x)$*, which, of course is convex. Let* $G_n^+(x)$ *[resp.* $G_n^-(x)$*] be the right (resp. left) derivatives of* $g_n(x)$*, and similarly for* $G^+(x),\ G^-(x)$*. Then, for all* $x \in I$*,*

$$\lim_{n\to\infty} \sup G_n^+(x) \leq G^+(x), \quad \lim_{n\to\infty} \sup G_n^-(x) \geq G^-(x). \tag{22}$$

In particular, if all the g_n *and* g *are differentiable at some point* $x \in I$*, then*

$$\lim_{n\to\infty} \frac{dg_n(x)}{dx} = \frac{dg(x)}{dx}. \tag{23}$$

Proof. Fix $x \in I$ and $x \pm y \in I$,

$$g_n(x+y) \geq g_n(x) + yG_n^+(x),$$

$$g_n(x-y) \geq g_n(x) - yG_n^-(x).$$

Fix y and take the limit $n \to \infty$. Then,

$$\lim_{n\to\infty} \sup G_n^+(x) \leq y^{-1}[g(x+y) - g(y)]$$

and similarly for $\lim_{n\to\infty} \inf G_n^-(x)$. Now let $y \downarrow 0$.

□

27. Final comments

1. Grand canonical approach. Bose mesoscopic systems (Bose gases confined in anharmonic and harmonic traps) admit to be studied, in the grand canonical ensemble by using standard mathematical approaches (by defining new effective intensive and extensive variables, for example "volume"). Recent experiments carried out in quasi uniform magnetic traps show good agreement with the theory developed for uniform Bose gases (textbook BEC). In this sense, far away from the critical regions, the grand canonical approach works fairly well. 2. Canonical and Microcanical ensembles: New theoretical strategies developed the last years, as the recently introduced "truncated Fock Space", gives us a profound insight on the behavior of such kind of mesoscopic systems, not only at a canonical ensemble level but also at the microscopic one. These approaches lead to a more exact and detailed description of the critical regions of phase transition than the obtained by appealing to the renormalization group theory, avoiding the infrared divergences of the standard thermodynamic limit and giving a suitable description of fluctuations close to the critical points. 3. These facts suggest that a complete theory connecting both, microscopic and mesoscopic levels should be possible for similar quantum many particle systems.

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